Digital Two-dimensional Bijective Reflection and Associated Rotation

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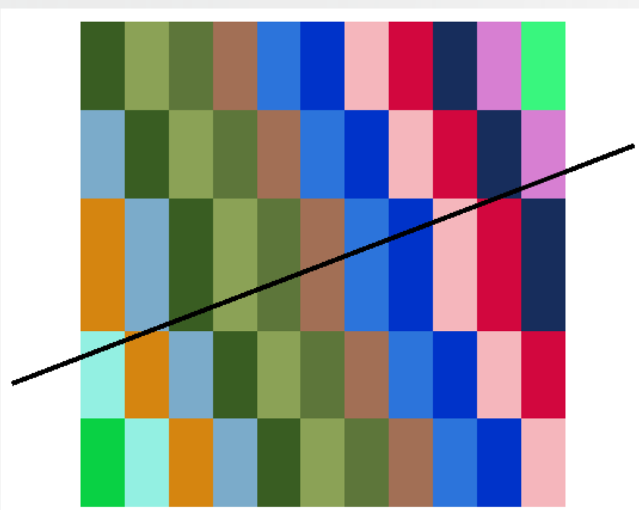
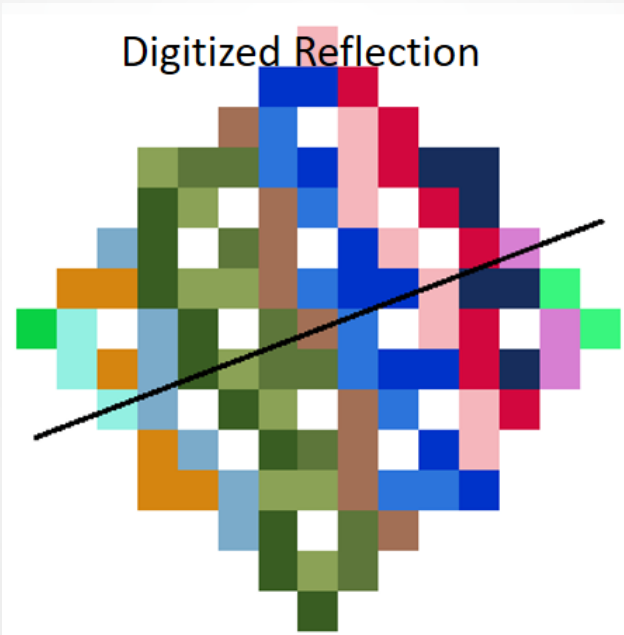
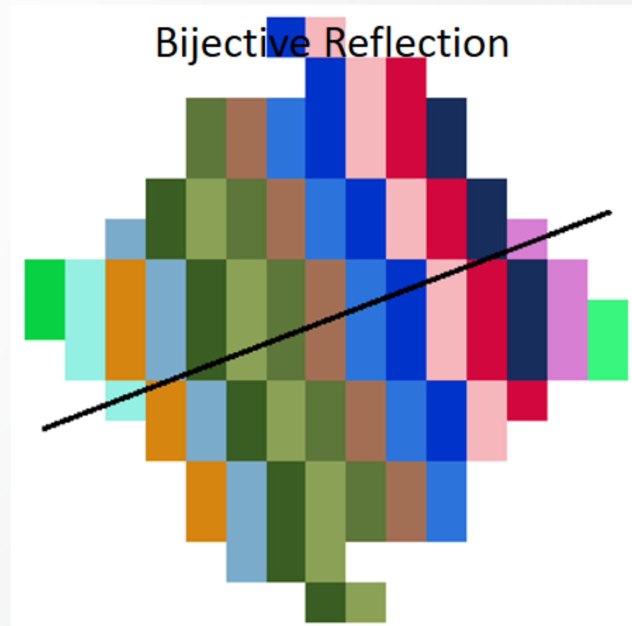
Goal :

- Propose a bijective Reflection Method for digital images

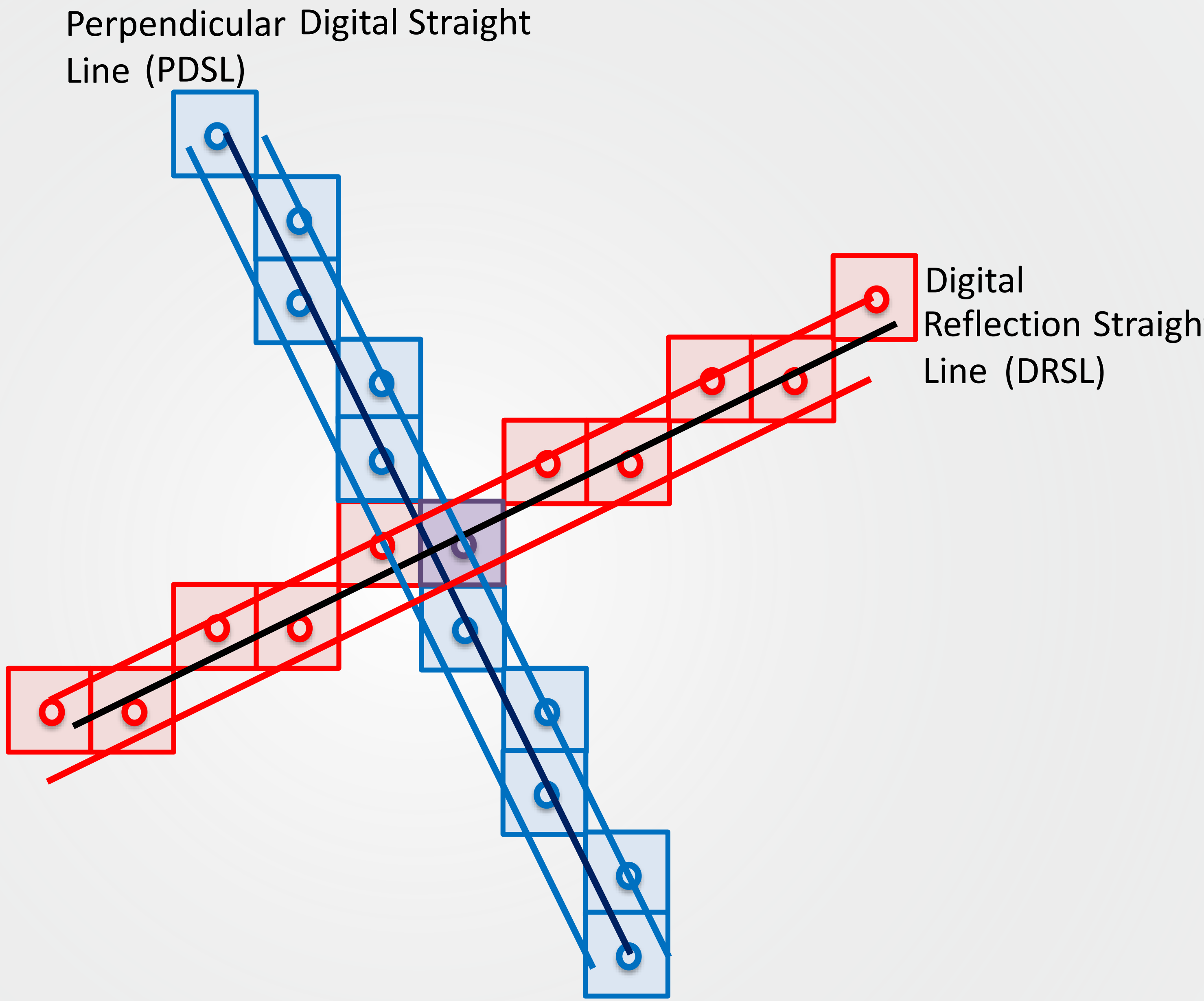
- Construct a bijective rotation method based on reflections





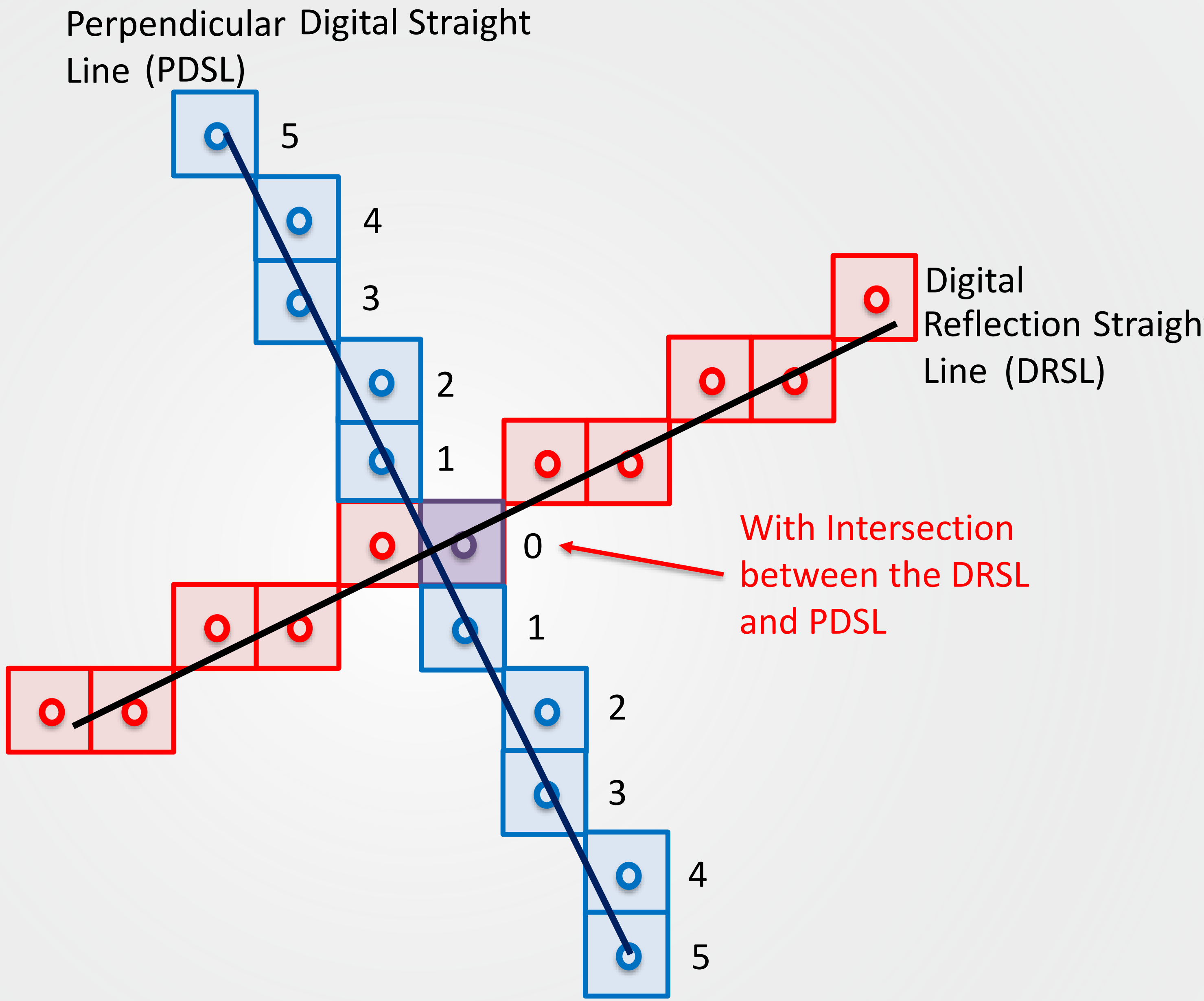


Idea :



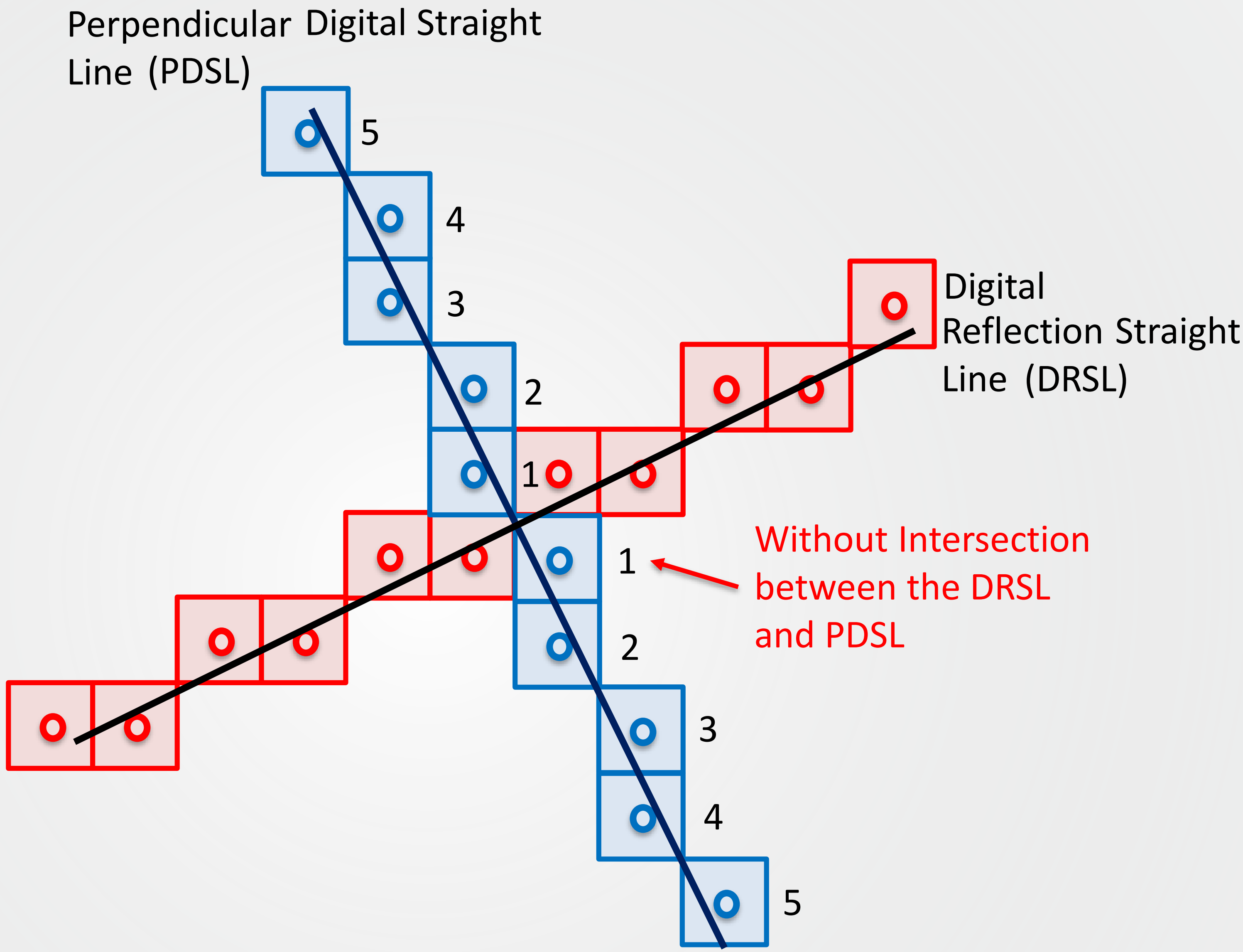
Perpendicular Digital Straight Line (PDSL)
Digital Reflection Straight Line (DRSL)

Idea :



Perpendicular Digital Straight Line (PDSL)
5
4
3
2
1
0
1
2
3
4
5
Digital Reflection Straight Line (DRSL)
With Intersection between the DRSL and PDSL

Idea :


Perpendicular Digital Straight Line (PDSL)
5
4
3
2
1
1
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3
4
5
Digital Reflection Straight Line (DRSL)
Without Intersection between the DRSL and PDSL

Mathematical details :

Continuous reflection Line $\mathcal{L}$: $C(x_o, y_o) \in \mathbb{R}^2$

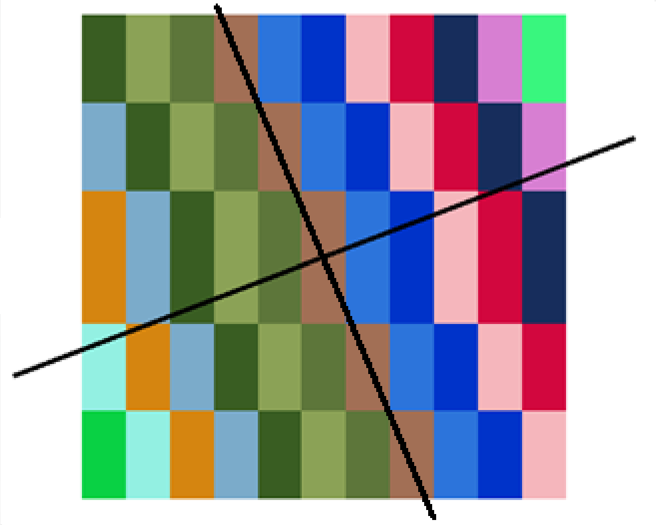
$$v = (b, a) = (\cos\theta, \sin\theta)$$

$$\mathcal{L} = \{(x, y) \in \mathbb{R}^2 : a(x - x_o) - b(y - y_o) = 0\}$$

Corresponding Naive Digital Straight Line

$$DRSL : -\frac{max(|a|, |b|)}{2} \leq a(x - x_o) - b(y - y_o) < \frac{max(|a|, |b|)}{2}$$

Mathematical details :



Perpendicular Line : $b(x - x_o) + a(y - y_o) = 0$

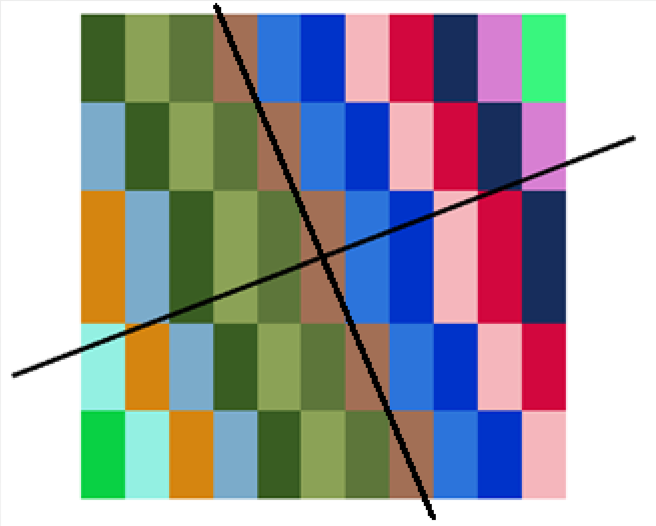
Perpendicular Digital Straight Line (PDSL) P_k :

$$\frac{(2k-1)\max(|a|,|b|)}{2} \leq b(x - x_o) + a(y - y_o) < \frac{\max(|a|,|b|)(2k+1)}{2}$$

This set of PDSLs partitions the digital space

Why Naive PDSLs ? Because there is a natural order on the points.

Mathematical details :



For a given (x,y) it is easy to determine the PDSL :

Perpendicular Digital Straight Line (PDSL) P_k :

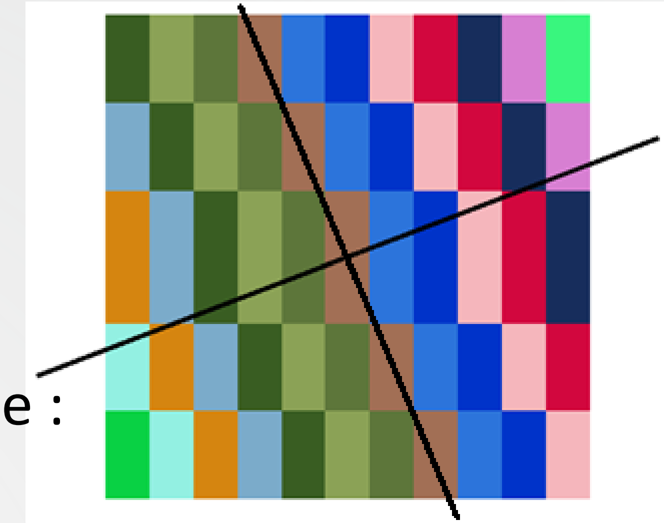
$$\frac{(2k-1)b}{2} \leq b(x-x_o)+a(y-y_o) < \frac{(2k+1)b}{2}$$

For a slope between -1 and 1

$$2k \leq 2\frac{b(x-x_o)+a(y-y_o)}{b}+1 < 2k+2$$

$$k = \left\lfloor (x-x_o)+\frac{a}{b}(y-y_o)+\frac{1}{2} \right\rfloor$$

Mathematical details :



For a given ordinate y, the abscissa x in Pk is unique :

Perpendicular Digital Straight Line (PDSL) P_k :

$$\frac{(2k-1)b}{2} \leq b(x - x_o) + a(y - y_o) < \frac{(2k+1)b}{2}$$

$$\frac{2k-1}{2} + x_o - \frac{a}{b}(y - y_o) \leq x < \frac{2k+1}{2} + x_o - \frac{a}{b}(y - y_o)$$

$$\mathcal{X}(y) = \left\lceil \frac{(2k-1)}{2} + x_o - (a/b)(y - y_o) \right\rceil$$

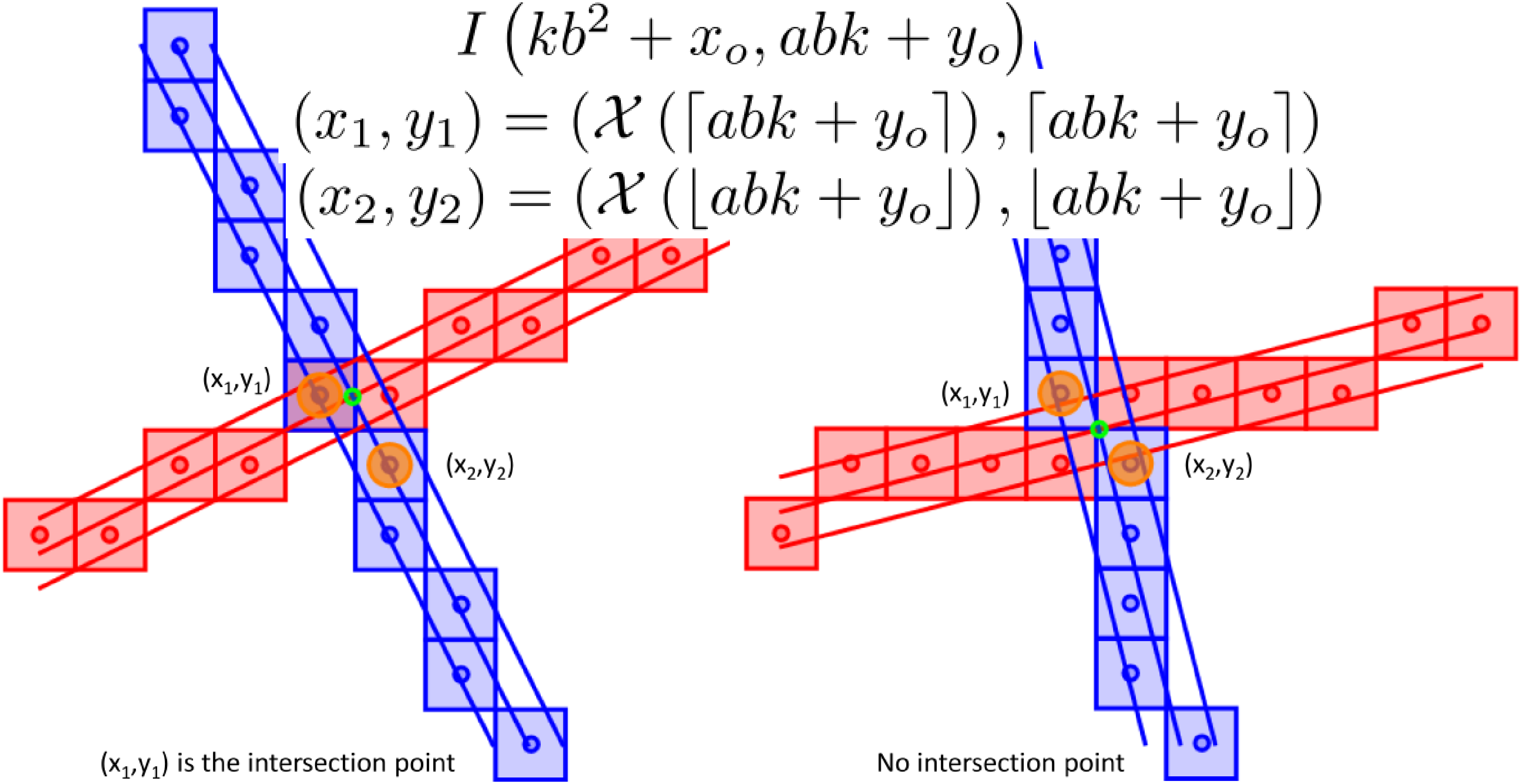
Mathematical details :

Last Question : at what condition does Pk intersect the DRSL ?

$$I\left(kb^2 + x_o, abk + y_o\right)$$

$$(x_1, y_1) = \left(\mathcal{X}\left(\lceil abk + y_o \rceil\right), \lceil abk + y_o \rceil\right)$$

$$(x_2, y_2) = \left(\mathcal{X}\left(\lfloor abk + y_o \rfloor\right), \lfloor abk + y_o \rfloor\right)$$



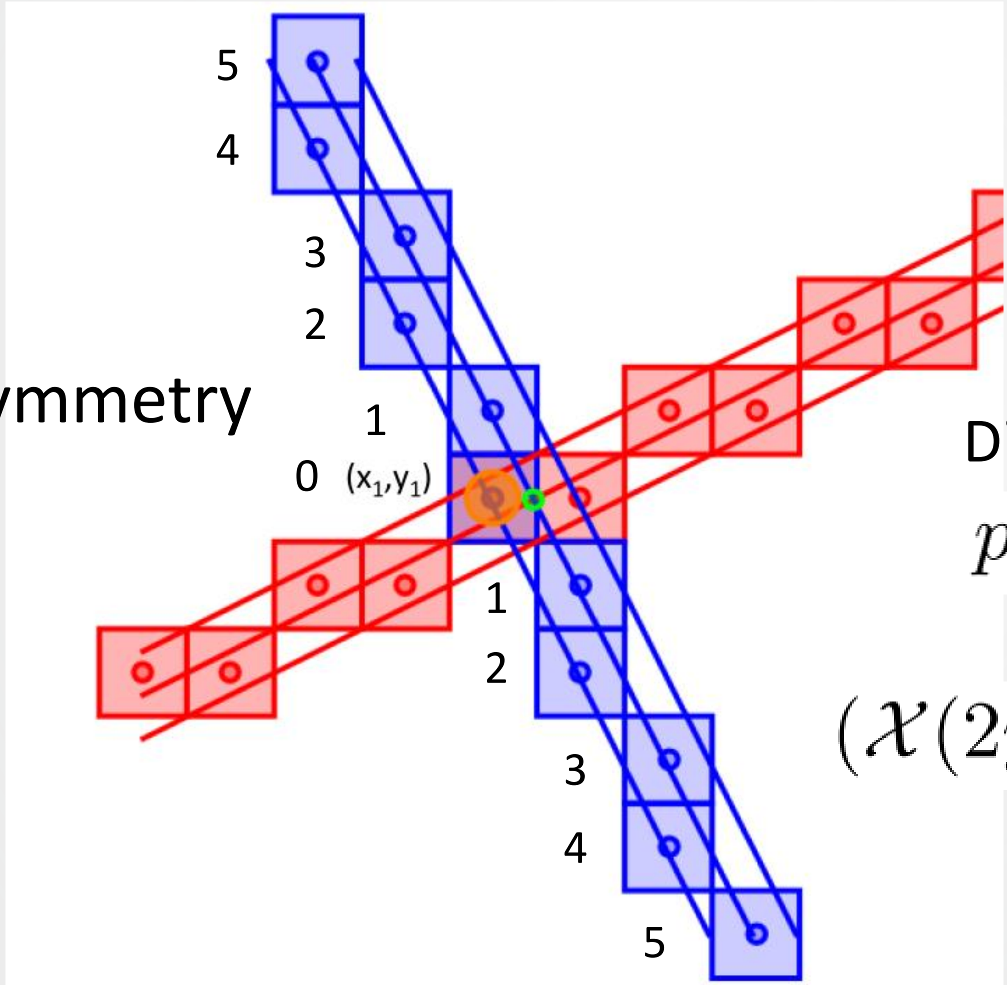
(x_1,y_1) is the intersection point

No intersection point

Mathematical details : (x_1,y_1) is the intersection point

$$(x_1, y_1) = (\mathcal{X}(\lceil abk + y_o \rceil), \lceil abk + y_o \rceil)$$

Vertical symmetry around y_1



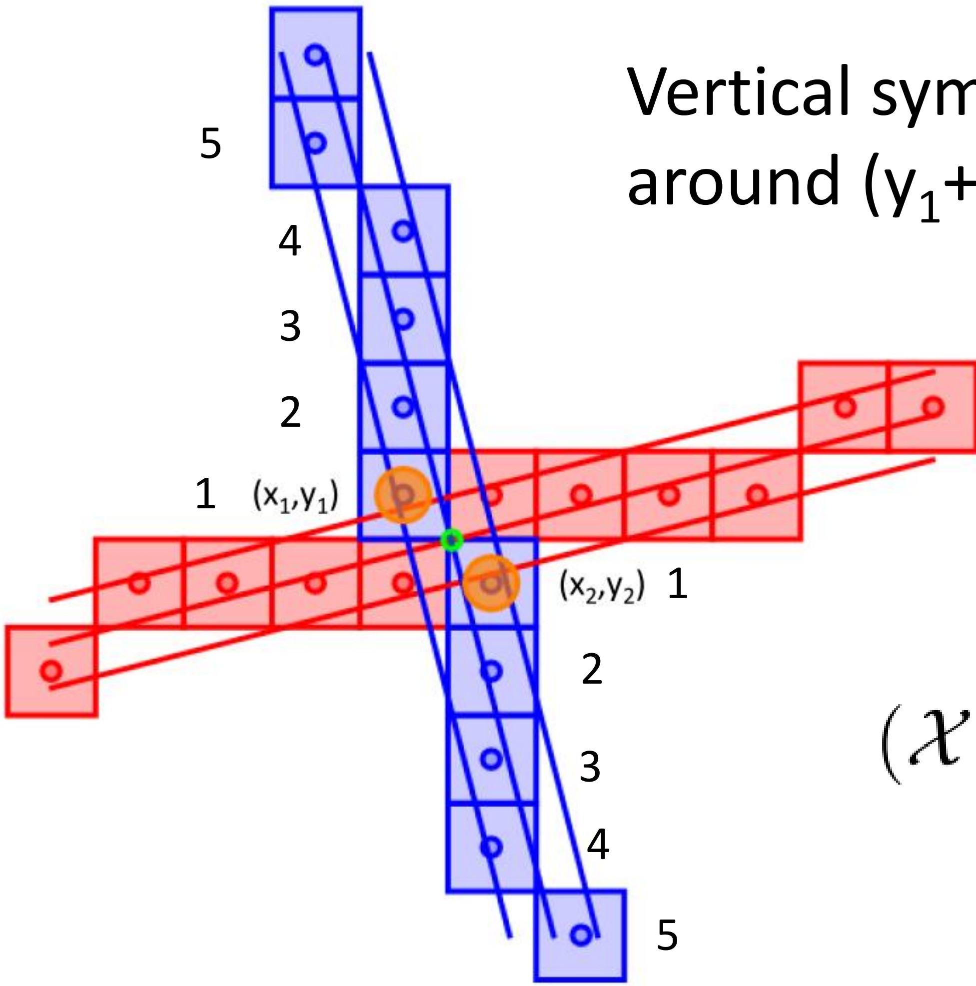
Digital Reflection of

$$p(x_p, y_p) \in P_k$$

$$(\mathcal{X}(2y_1 - y_p), 2y_1 - y_p)$$

Mathematical details : No Intersection point

Vertical symmetry around $(y_1+y_2)/2$



Digital Reflection of

$$p\,(x_p, y_p)\;\in\;P_k$$

$$(\mathcal{X}(y_1 + y_2 - y_p), y_1 + y_2 - y_p)$$

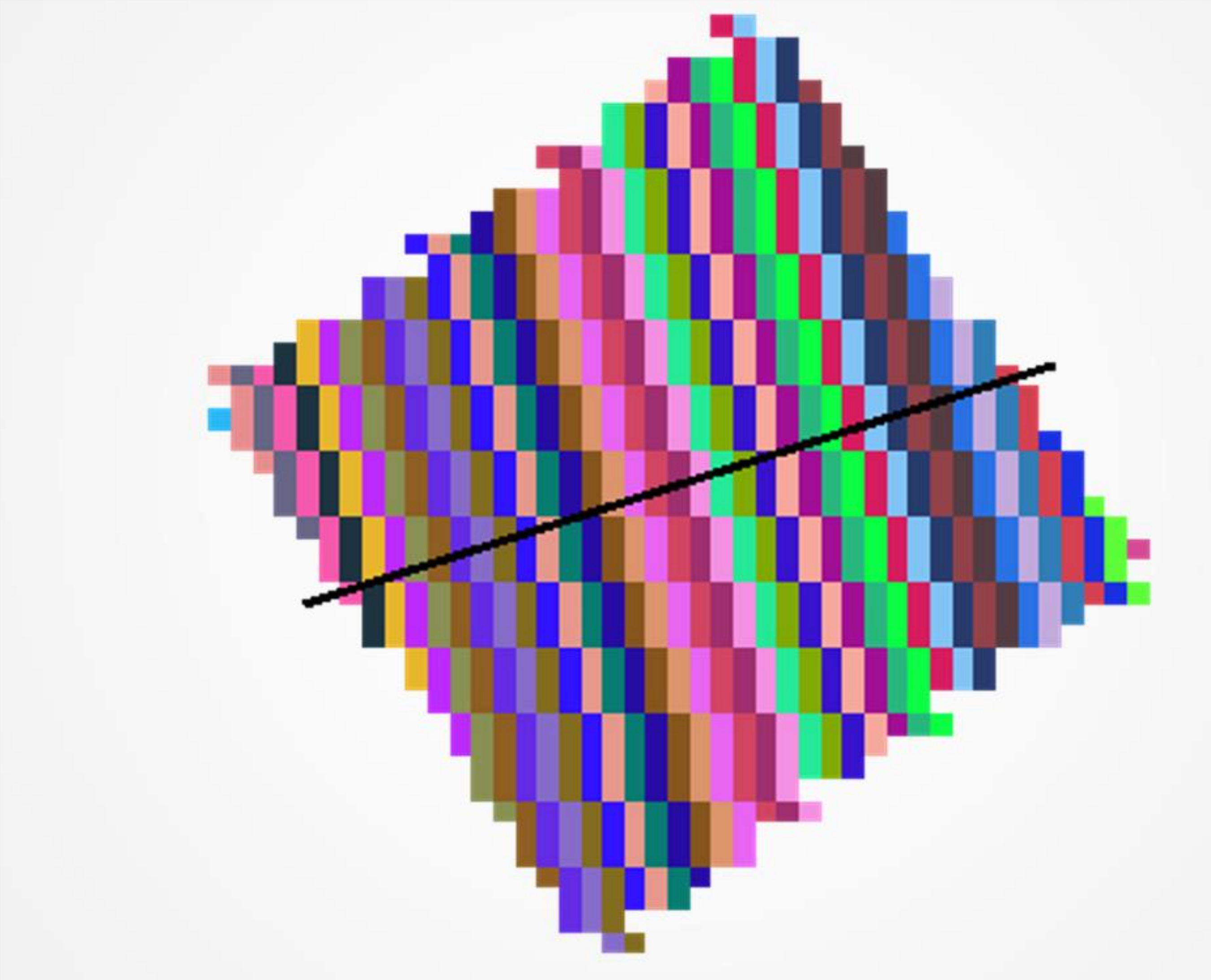
Bijective Digital Reflection Algorithm:

Algorithm 1: REFLECTION TRANSFORM $R_{\theta,(x_o,y_o)}$

Input : $(x,y) \in \mathbb{Z}^2, (x_o,y_o) \in \mathbb{R}^2, -\pi/4 \leqslant \theta < \pi/4$
Output: $(x',y') \in \mathbb{Z}^2$

$k = \left\lfloor (x - x_o) + \frac{a}{b}(y - y_o) + \frac{1}{2} \right\rfloor$
Function $\mathcal{X}(y) : \mathbb{Z} \mapsto \mathbb{Z} : \mathcal{X}(y) = \left\lceil \frac{(2k-1)}{2} + x_o - (a/b)(y - y_o) \right\rceil$
$(x_1, y_1) = (\mathcal{X}(\lceil abk + y_o \rceil), \lceil abk + y_o \rceil)$
$(x_2, y_2) = (\mathcal{X}(\lfloor abk + y_o \rfloor), \lfloor abk + y_o \rfloor)$
If $-b/2 \leqslant a(x_1 - x_o) - b(y_1 - y_o) < b/2$ Then
    $(x',y') = (\mathcal{X}(2y_1 - y), 2y_1 - y)$
Elseif $-b/2 \leqslant a(x_2 - x_o) - b(y_2 - y_o) < b/2$ Then
    $(x',y') = (\mathcal{X}(2y_2 - y), 2y_2 - y)$
Else $(x',y') = (\mathcal{X}(y_1 + y_2 - y), y_1 + y_2 - y)$
return (x',y')

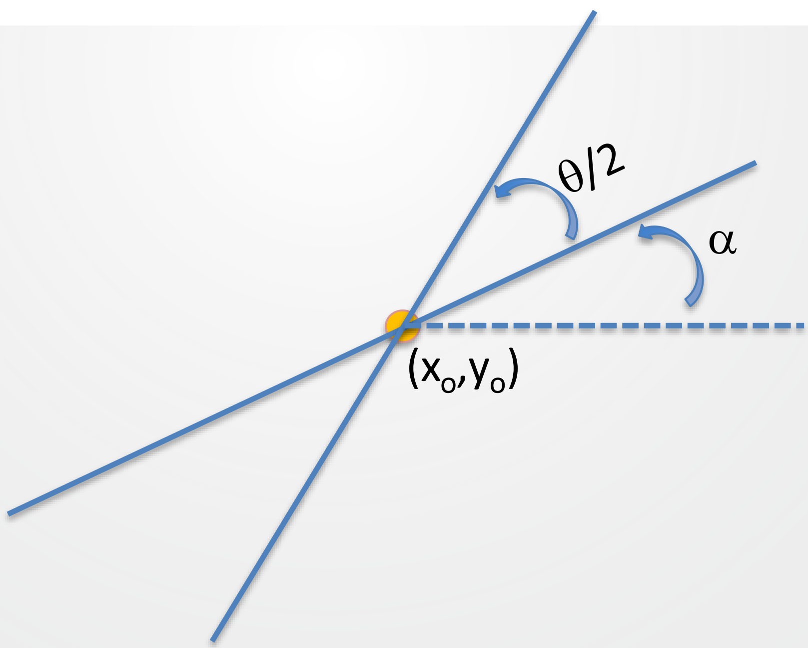
Bijective Digital Reflection Algorithm:



Bijective Digital Rotation based on Reflections:

Digital Bijective Rotation based on two Digital Bijective Reflections:

$$Rot_{\theta,(x_o,y_o)}(x,y) = \left(R_{\alpha+\frac{\theta}{2},(x_o,y_o)} \circ R_{\alpha,(x_o,y_o)}\right)(x,y)$$



Bijective Digital Rotation based on Reflections:

Digital Bijective Rotation based on two Digital Bijective Reflections:

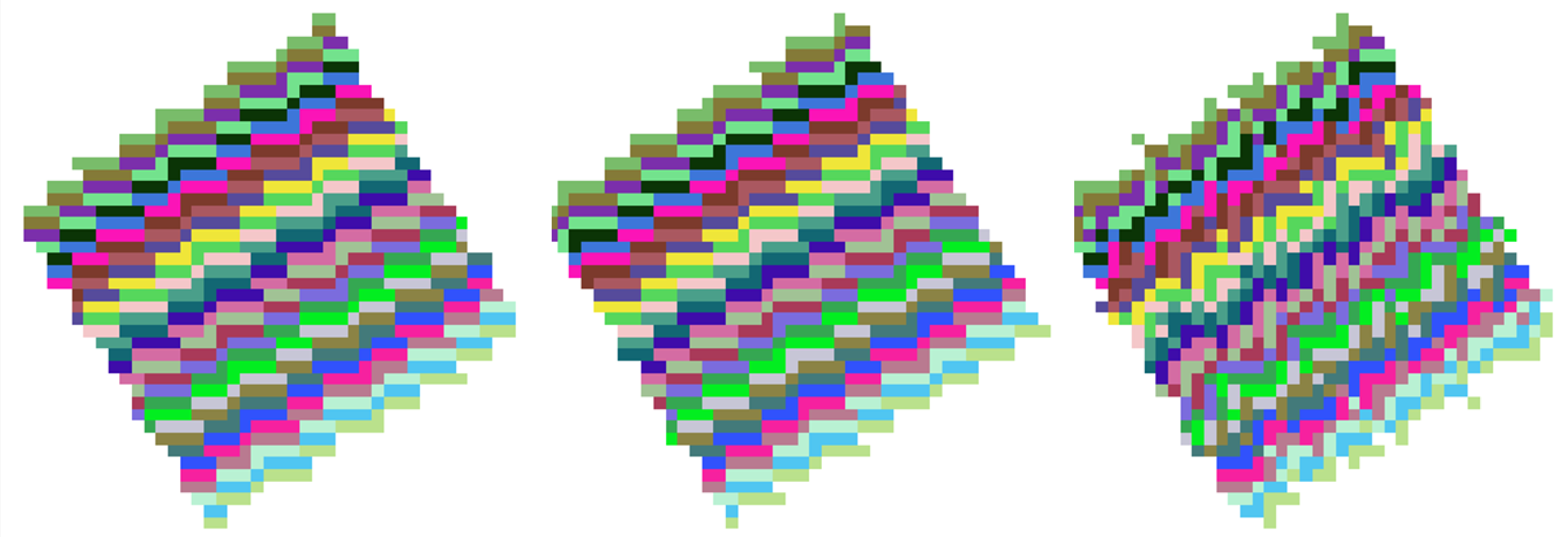
$$Rot_{\theta,(x_o,y_o)}(x,y) = \left(R_{\alpha+\frac{\theta}{2},(x_o,y_o)} \circ R_{\alpha,(x_o,y_o)}\right)(x,y)$$

Characteristics :

- Bijective
- Arbitrary Rotation Center
- Arbitrary Rotation Angle
- Easily Invertible

There is however an extra parameter α that needs to be assessed.

Bijective Digital Rotation based on reflections:



$(x_0,y_0) = (0,0)$
$\alpha = 0$

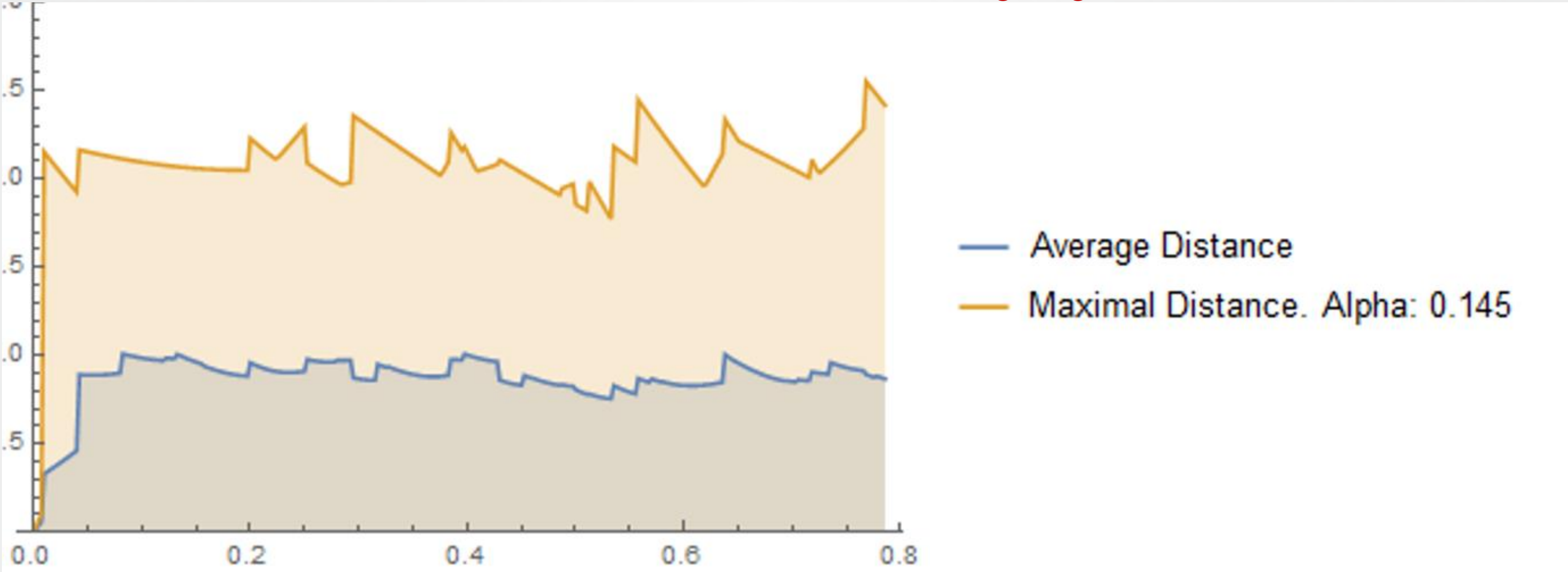
$(x_0,y_0) = (0.25,0.25)$
$\alpha = 0$

$(x_0,y_0) = (0,0)$
$\alpha = \pi/8$

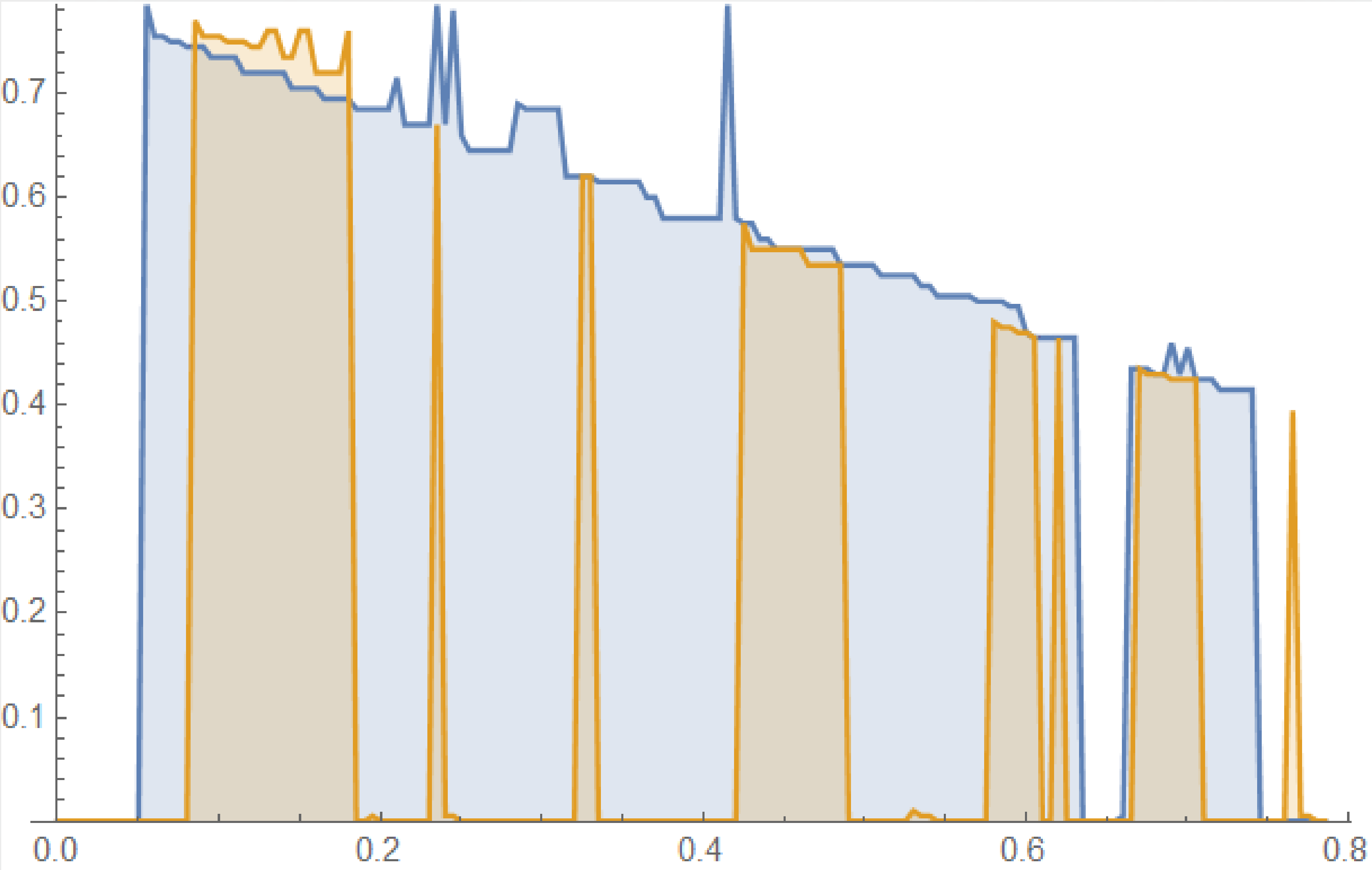
Rotation Evaluation:

Compute the **average** and **maximal** distance between the **Continuous** and **digital** transform of image points

Influence of the angle α (here $x_o=y_o=0$)



Rotation Evaluation: α with minimal error



Rotation Evaluation and General Conclusion:

- It seems to be a good idea to keep α=0 for the sake of simplicity but more work is required.
  x_o doesn't seem to matter, y_o does.

- The method fares worse than the equivalent Shear based Rotation. It is however easier to set up;

- First time, to the authors best knowledge, that a bijective digital reflection transform has been proposed;

- Rotations in high dimensions are defined by reflections. The main interest of this rotation method seems to be its extension to higher dimensions.