A Unified Topological Framework for Digital Imaging

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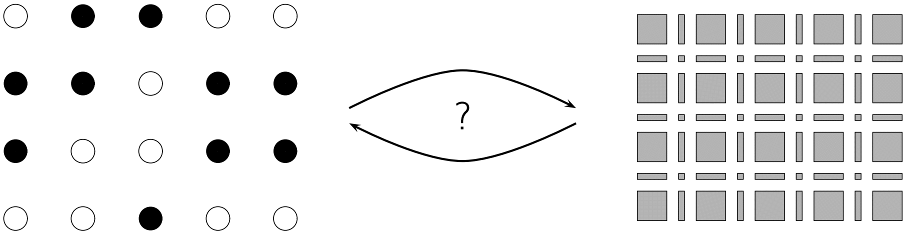
Presentation outline

Embedding



$\mathbb{Z}^2$ $\mathbb{F}^2$

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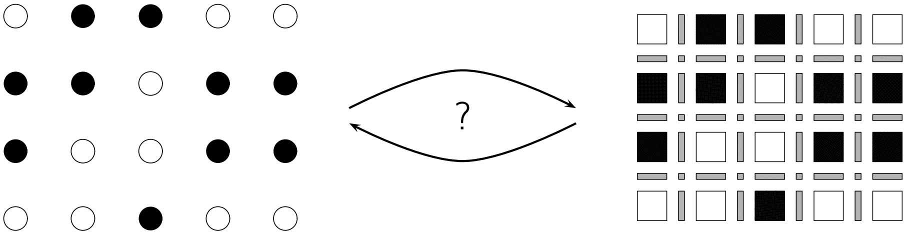
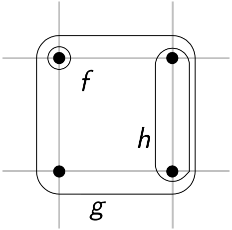
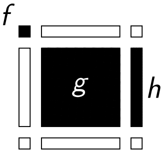


$\mathbb{Z}^2$

$\mathbb{F}^2$

$\mathbb{F}^n$ a discrete topological space

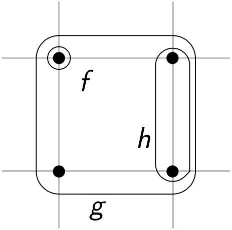


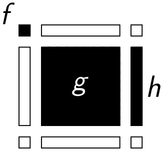


k-face: set of 2^k points of $\mathbb{Z}^n$ forming a unit cube.

- $(\mathbb{F}^n, \subseteq)$ is a POSET
- $\Rightarrow$ $\mathbb{F}^n$ has a natural topology where a subspace $\{f, g\}$ is connected iff f and g are comparable.

$\mathbb{F}^n$ a discrete topological space





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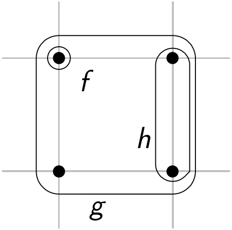

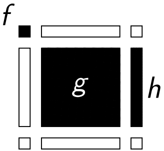

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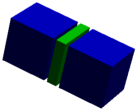
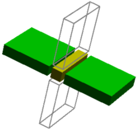
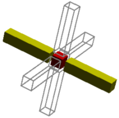
Opposite faces, connectivity function

- Let f be a k-face. Two $(k+1)$-faces a, b are opposite w.r.t. f if $a \cap b = f$ and there is no face in $\mathbb{F}^n$ including $a \cup b$.







We set $\text{opp}(f) = \{\{a, b\} \mid a \text{ is opposite to } b \text{ w.r.t. } f\}$

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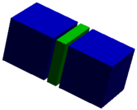 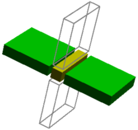 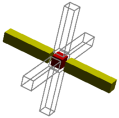

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Regular images

- Let $\varepsilon : [1, n] \rightarrow \{-1, 1\}$ be a function called connectivity function. A function $\mu : \mathbb{F}^n \rightarrow \{0, 1\}$ is an ε-regular image if for all m-face $f \in \mathbb{F}^n$, $m \in [1, n-1]$, we have, recursively,

$$\mu(f) = \begin{cases} \bigvee_{\{a,b\} \in \mathrm{opp}(f)} \mu(a) \wedge \mu(b) & \text{if } \varepsilon(m+1) = +1 \\ \bigwedge_{\{a,b\} \in \mathrm{opp}(f)} \mu(a) \vee \mu(b) & \text{if } \varepsilon(m+1) = -1 \end{cases}$$

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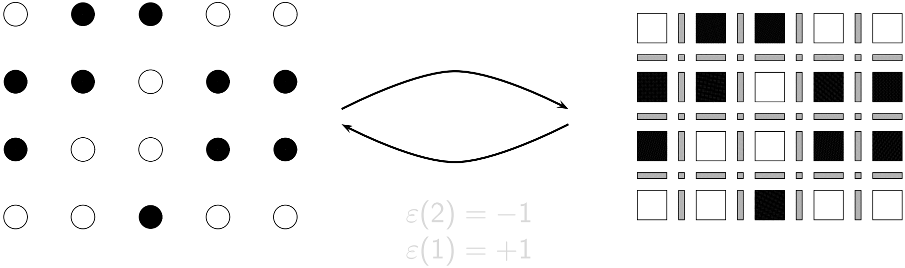
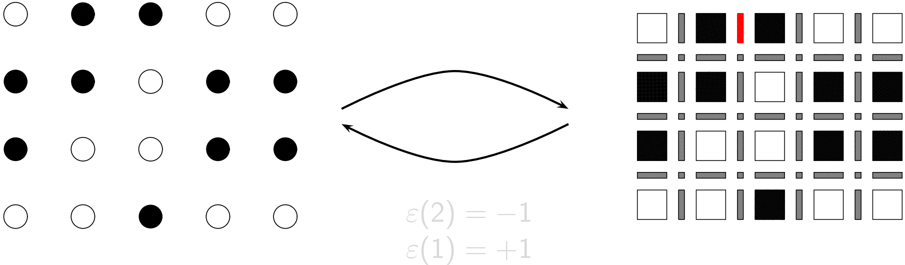


Examples - 1



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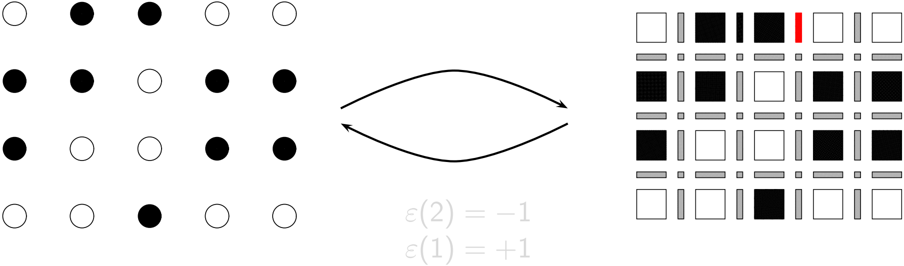

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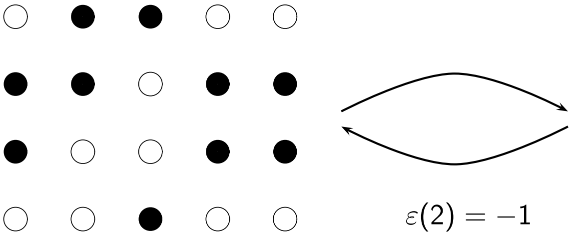
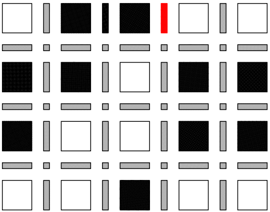
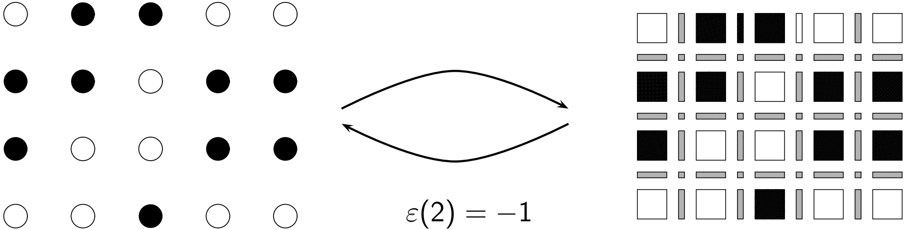


$\varepsilon(2) = -1$

$\varepsilon(1) = +1$





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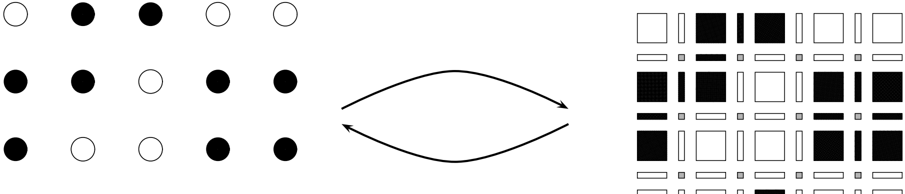

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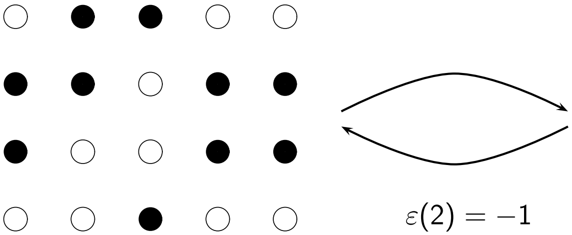

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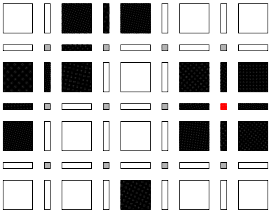

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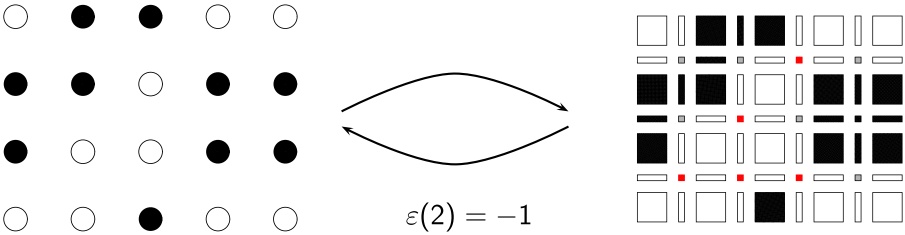

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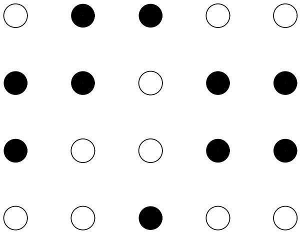

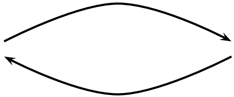

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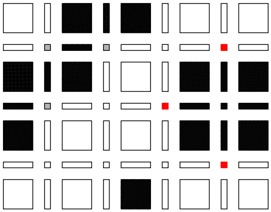

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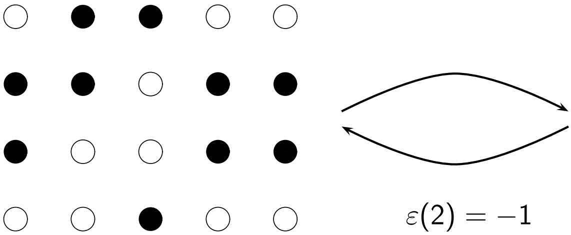

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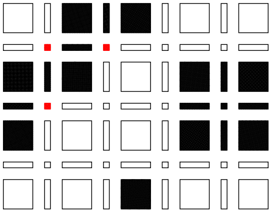

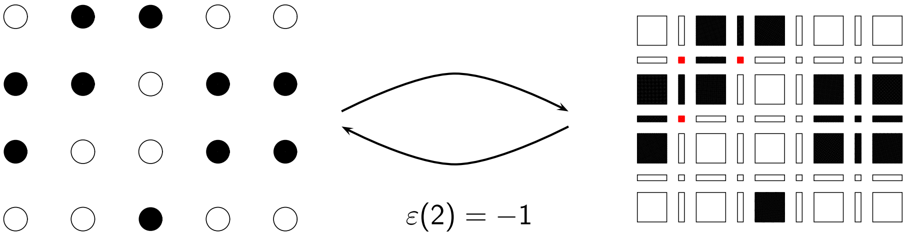

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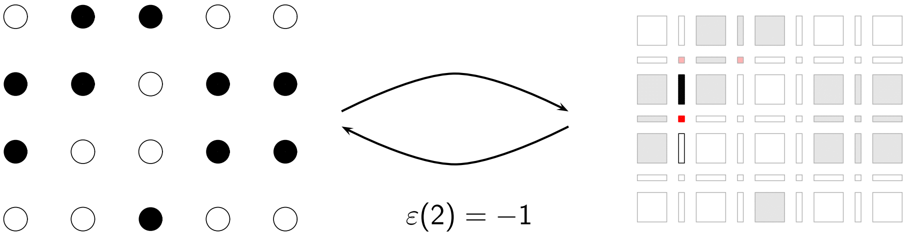

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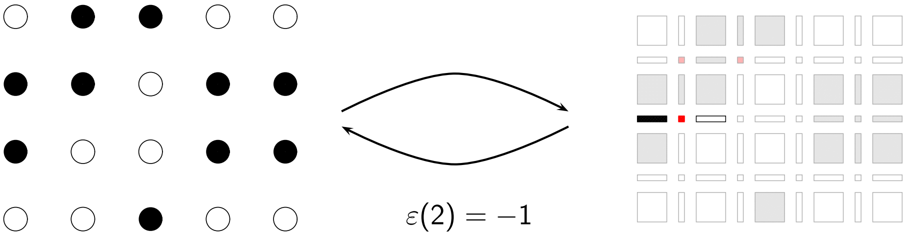

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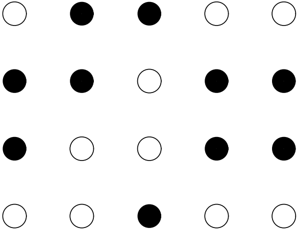

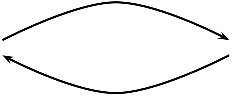

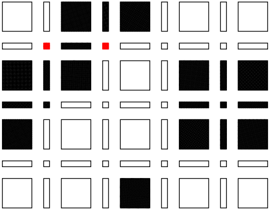

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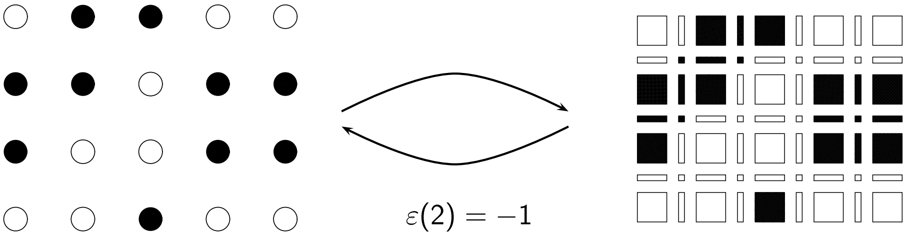

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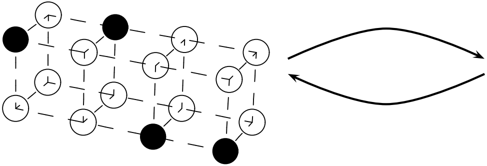
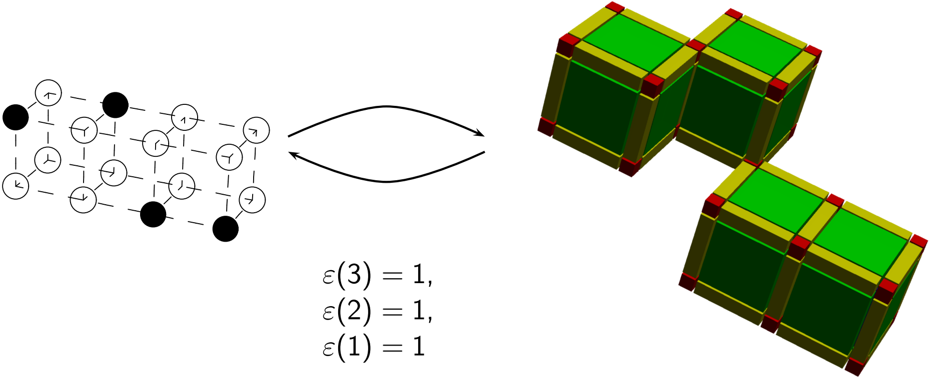
Examples - 2



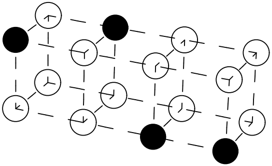
$\varepsilon = \ldots$

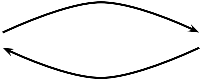
Examples - 2



$$\varepsilon(3) = 1,$$
$$\varepsilon(2) = 1,$$
$$\varepsilon(1) = 1$$

Examples - 2





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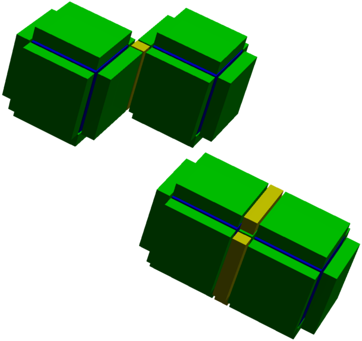

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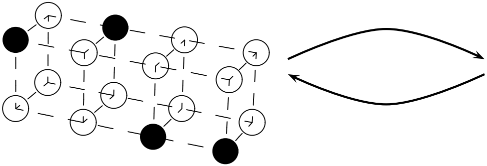

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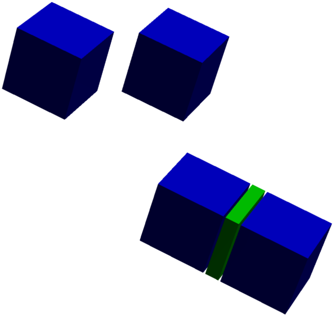

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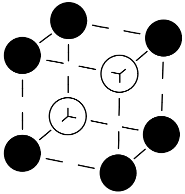

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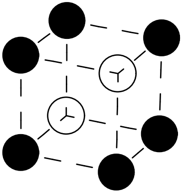
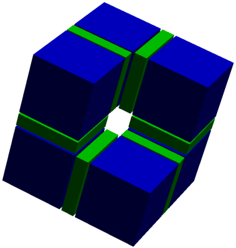
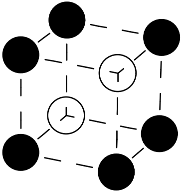
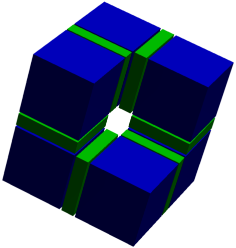




$$\varepsilon(3) = \varepsilon(2) = \varepsilon(1) = -1$$

Examples - 3





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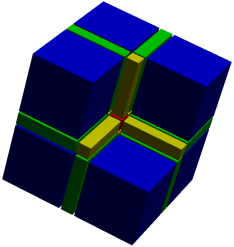


$\varepsilon(3) = \varepsilon(1) = -1,\ \varepsilon(2) = 1$

Computing k-faces from facets (1): ε constant

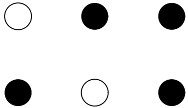
Adjacencies in $\mathbb{Z}^n$	Connectivity function	#facets
$(2n,\ 3^n - 1)$	-1	2^{n-k} (all)
$(3^n - 1,\ 2n)$	$+1$	1

Black faces : minimal number of black facets in the neighborhood.

		#facets		
Adjacencies in $\mathbb{Z}^3$	ε	$k=0$	$k=1$	$k=2$
$(6, 18)$	$1 \to -1$ $2 \to +1$ $3 \to -1$	6	3	2
$(18, 6)$	$1 \to +1$ $2 \to -1$ $3 \to +1$	3	2	1

Black faces : minimal number of black facets in the neighborhood.

Duality



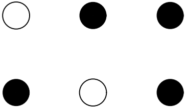
$\overset{\neg}{\longrightarrow}$

$\downarrow \varepsilon : \begin{cases} 1 \to -1 \\ 2 \to +1 \end{cases}$

$\downarrow (-\varepsilon) : \begin{cases} 1 \to -1 \\ 2 \to +1 \end{cases}$

$\overset{\neg}{\longrightarrow}$

Duality



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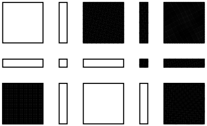

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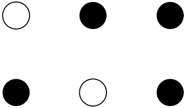

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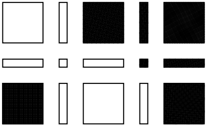

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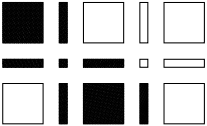

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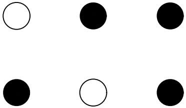

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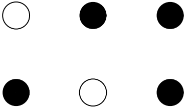

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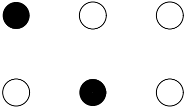
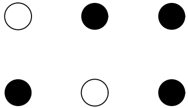


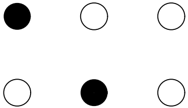
$$\Big\downarrow \; \varepsilon : \begin{cases} 1 \to -1 \\ 2 \to +1 \end{cases} \qquad\qquad \Big\downarrow \; (-\varepsilon) : \begin{cases} 1 \to -1 \\ 2 \to +1 \end{cases}$$

$$\xrightarrow{\neg}$$

Duality



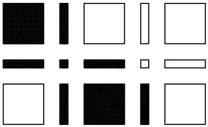
$$\xrightarrow{\neg}$$



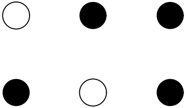
$$\downarrow \varepsilon : \begin{cases} 1 \to -1 \\ 2 \to +1 \end{cases}$$

$$\downarrow (-\varepsilon) : \begin{cases} 1 \to -1 \\ 2 \to +1 \end{cases}$$

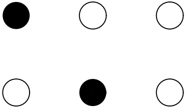
$$\xrightarrow{\neg}$$



Duality

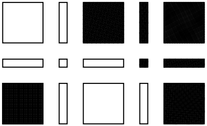


$\xrightarrow{\neg}$

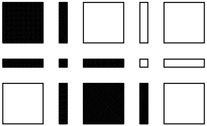


$\Big\downarrow \; \varepsilon : \begin{cases} 1 \to -1 \\ 2 \to +1 \end{cases}$

$\Big\downarrow \; (-\varepsilon) : \begin{cases} 1 \to -1 \\ 2 \to +1 \end{cases}$



$\xrightarrow{\neg}$



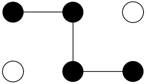
No artifact:

Let ε be a connectivity function.
Let $\mu : \mathbb{F}^n \to \{0, 1\}$ be an ε-regular image. Let $x \in \{0, 1\}$.
The interior of $\mu^{-1}(\{x\})$ is a regular open set.
The closure of $\mu^{-1}(\{x\})$ is a regular closed set.

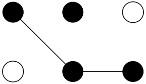
Paths: $\mathbb{Z}^n \to \mathbb{F}^n$

Path in $\mathbb{Z}^n$: sequence of α-adjacent points.

Path in $\mathbb{F}^n$: sequence of comparable faces.



$\alpha = 4$

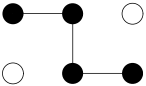


$\alpha = 8$

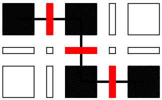
Paths: $\mathbb{Z}^n \to \mathbb{F}^n$

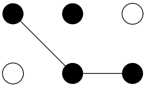
Path in $\mathbb{Z}^n$: sequence of α-adjacent points.

Path in $\mathbb{F}^n$: sequence of comparable faces.

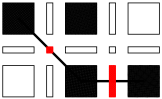


$\alpha = 4$



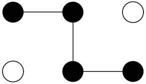


$\alpha = 8$

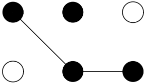


Paths: $\mathbb{Z}^n \to \mathbb{F}^n$

Path in $\mathbb{Z}^n$: sequence of α-adjacent points.

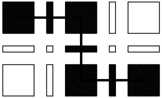


$\alpha = 4$

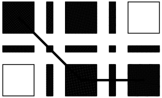


$\alpha = 8$

Path in $\mathbb{F}^n$: sequence of comparable faces.



$\varepsilon = -1$

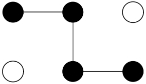


$\varepsilon = +1$

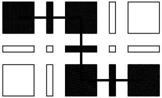
Paths: $\mathbb{Z}^n \rightarrow \mathbb{F}^n$

Path in $\mathbb{Z}^n$: sequence of α-adjacent points.

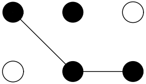
Path in $\mathbb{F}^n$: sequence of comparable faces.



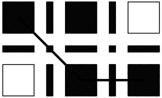
$\alpha = 4$



$\varepsilon = -1$

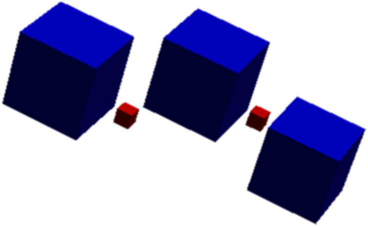


$\alpha = 8$



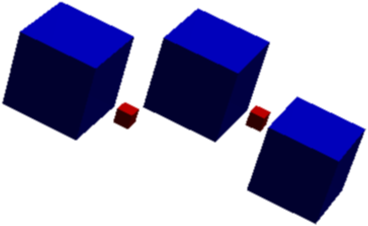
$\varepsilon = +1$

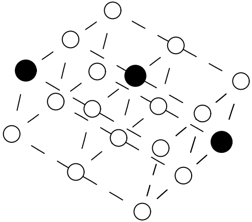
Connected in $\mathbb{Z}^n \Rightarrow$ Connected in $\mathbb{F}^n$



$\Rightarrow$ One-to-one correspondence between the connected components (object and background)

Paths: $\mathbb{F}^n \rightarrow \mathbb{Z}^n$

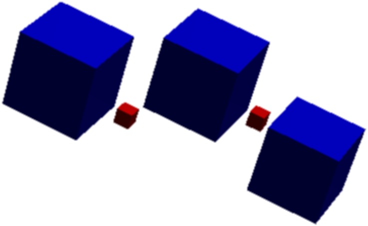


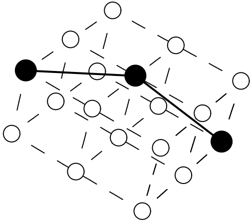


Connected ?

$\Rightarrow$ One-to-one correspondence between the connected components (object and background)

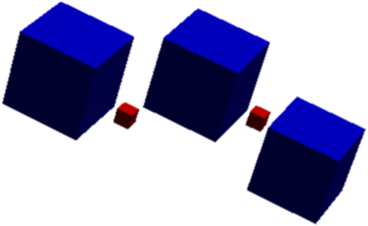
Paths: $\mathbb{F}^n \to \mathbb{Z}^n$

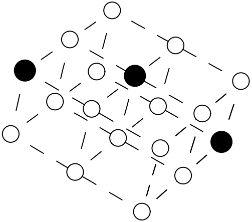




$\alpha = 26$

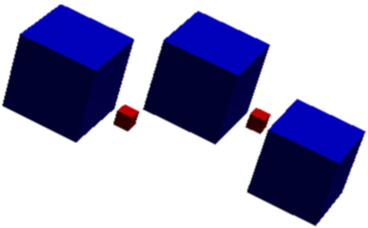
$\Rightarrow$ One-to-one correspondence between the connected components (object and background)

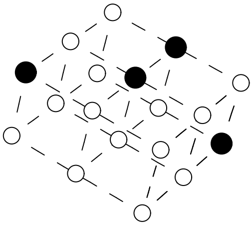




$\alpha = 18$

$\Rightarrow$ One-to-one correspondence between the connected components (object and background)

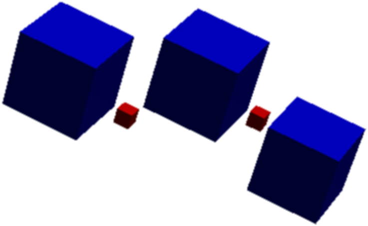


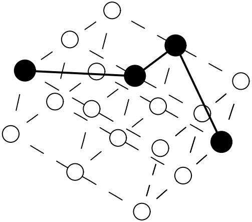


$\alpha = 18$

$\Rightarrow$ One-to-one correspondence between the connected components (object and background)

Paths: $\mathbb{F}^n \to \mathbb{Z}^n$

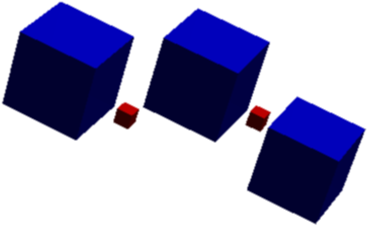


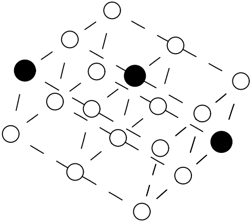


$\alpha = 18$

$\Rightarrow$ One-to-one correspondence between the connected components (object and background)

Paths: $\mathbb{F}^n \to \mathbb{Z}^n$

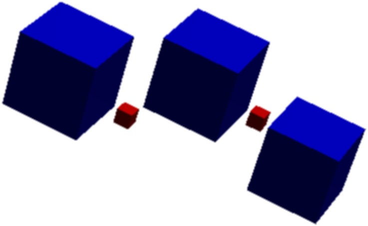


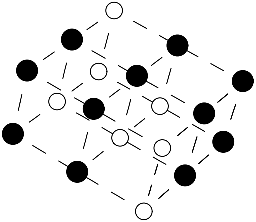


$\alpha = 6$

$\Rightarrow$ One-to-one correspondence between the connected components (object and background)

Paths: $\mathbb{F}^n \rightarrow \mathbb{Z}^n$

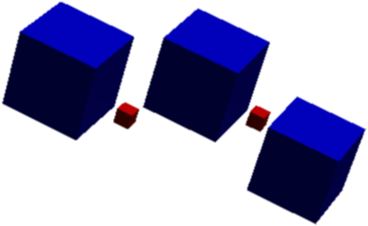


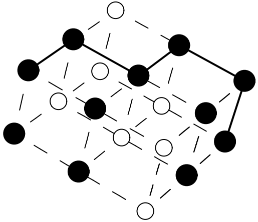


$\alpha = 6$

$\Rightarrow$ One-to-one correspondence between the connected components (object and background)

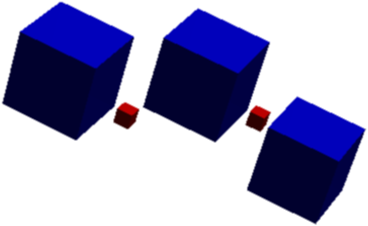
Paths: $\mathbb{F}^n \to \mathbb{Z}^n$

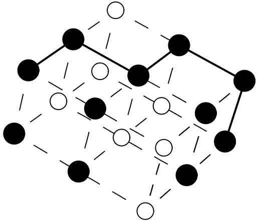




$\alpha = 6$

$\Rightarrow$ One-to-one correspondence between the connected components (object and background)





$\alpha = 6$

$\Rightarrow$ One-to-one correspondence between the connected components (object and background)

Digital fundamental group ⇄ $\mathbb{F}^n$ path fundamental group

isomorphism

$\mathbb{F}^n$ continuous path fundamental group

Digital fundamental group ⇄ isomorphism ⇄ $\mathbb{F}^n$ path fundamental group

$\mathbb{F}^n$ path fundamental group ⇅ isomorphism ⇅ $\mathbb{F}^n$ continuous path fundamental group

Salient outcomes

$$(\mathbb{Z}^n, \alpha, \beta) \quad \rightleftarrows \quad \mathbb{F}^n$$

$$\varepsilon = f(\alpha, \beta)$$

- connected components are in one-to-one correspondence
- groups of "loops", up to deformations, are isomorphic
- easy to compute
- easily extensible to other codomains than $\{0, 1\}$

Salient outcomes

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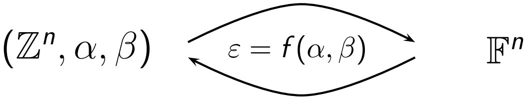

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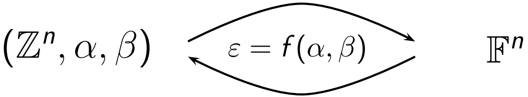

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Thank you