Some morphological operators on complex spaces

Fábio Dias Jean Cousty Laurent Najman

Université Paris-Est, Laboratoire d'Informatique Gaspard-Monge, Équipe A3SI, ESIEE

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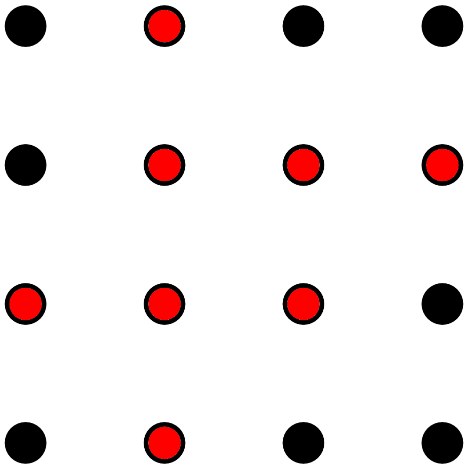
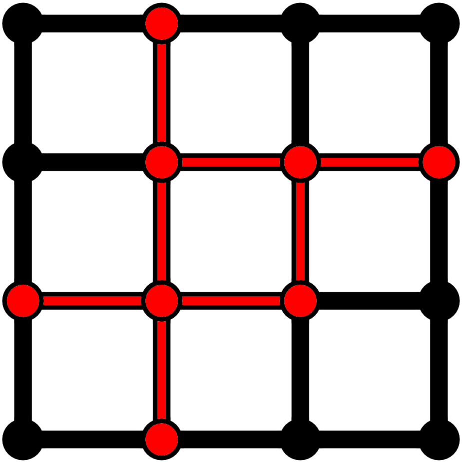
diasf@esiee.fr

Outline

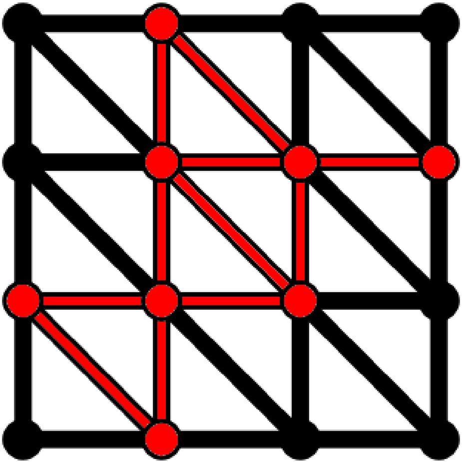
(Binary) Images



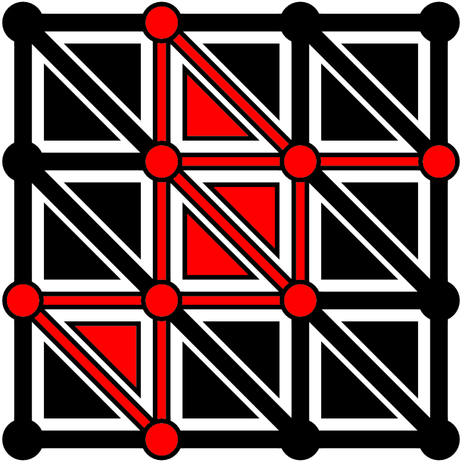
Graphs and images



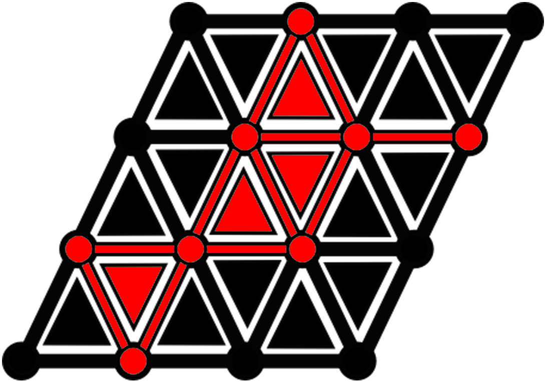
Graphs and images



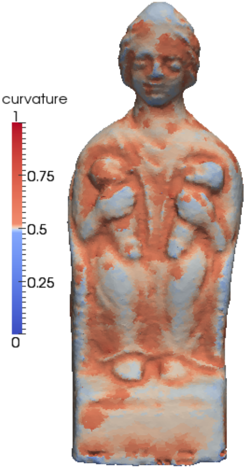
Simplicial complexes and images



Simplicial complexes and images



Meshes



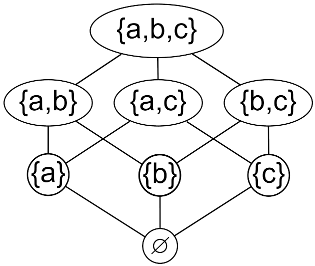
Mathematical Morphology and lattices

Lattice:

A *lattice* is a partially ordered set, that also have a least upper bound, called *supremum*, and a greatest lower bound, called *infimum*.

Example: the power set of $\{a, b, c\}$:

- Supremum: union.
- Infimum: intersection.



Adjunctions

Two operators

$$\alpha : \quad \mathcal{L}_2 \to \mathcal{L}_1 \tag{1}$$

$$\alpha^A : \quad \mathcal{L}_1 \to \mathcal{L}_2 \tag{2}$$

form an *adjunction* (α^A, α) if :

$$\alpha(a) \leq_1 b \leftrightarrow a \leq_2 \alpha^A(b) \tag{3}$$

Property:

- α is a dilation: commutes with the supremum.
- α^A is an erosion: commutes with the infimum.

Filters

A *filter* is an operator $\beta : \mathcal{L} \to \mathcal{L}$ that is:

- Increasing: $X \subseteq Y \implies \beta(X) \subseteq \beta(Y)$
- Idempotent: $\beta(\beta(X)) = \beta(X)$.

Closing $\phi : \mathcal{L} \to \mathcal{L}$

Extensive: $X \subseteq \phi(X)$

Opening $\gamma : \mathcal{L} \to \mathcal{L}$

Anti-extensive: $\gamma(X) \subseteq X$

Granulometry

Granulometric property:

$\forall d_1, d_2 \in \mathbb{N}$, if $d_1 \leq d_2$:

- $\gamma_{d_2}(X) \subseteq \gamma_{d_1}(X)$
- $\phi_{d_1}(X) \subseteq \phi_{d_2}(X)$

The families of operators $\{\gamma_d \mid d \in \mathbb{N}\}$ and $\{\phi_d \mid d \in \mathbb{N}\}$ are granulometries.

Objective

Investigate morphological operators acting on complexes:

- Dilations,
- Erosions,
- Openings,
- Closings,
- Granulometries...

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Investigate morphological operators acting on complexes:

- Dilations,
- Erosions,
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- Closings,
- Granulometries...

To achieve:

- "Smaller resolution".
- Better filtering results on images/meshes.

Simplex

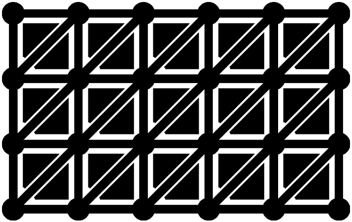
Basic definitions

- *Simplex*: any nonempty set.
- *Dimension* of a simplex: $|x| - 1$.

Example of simplices $\{a\}$, $\{a, b\}$ and $\{a, b, c\}$:

Complexes and subcomplexes

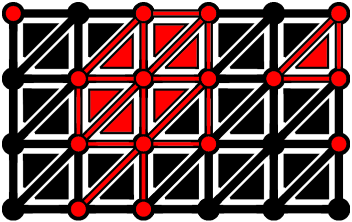
Complex: Any set X such that, for any $x \in X$, all nonempty subsets of x also belongs to X.



$\mathbb{C}$: a nonempty complex, of dimension n.

Complexes and subcomplexes

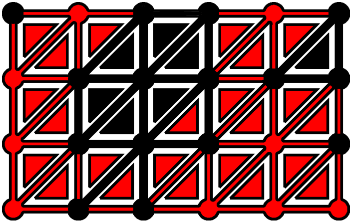
Subcomplex: A subset Y of a complex X such that Y is also a complex.



$\mathcal{C}$: the set of all subcomplexes of $\mathbb{C}$.

Complexes and subcomplexes

Star: The complement of a subcomplex.



$\mathcal{S}$: the set of all stars of $\mathbb{C}$.

Lattices of interest:

The set $\mathcal{C}$ is a lattice:

- Closed under union and intersection,
- Set inclusion as order relation.

The lattice $\mathcal{C}$ is **not** complemented.

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- Closed under union and intersection,
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- Set inclusion as order relation.

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Lattices of interest:

The set $\mathcal{C}$ is a lattice:

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The lattice $\mathcal{C}$ is **not** complemented.

The set $\mathcal{S}$ is a lattice:

- Closed under union and intersection,
- Set inclusion as order relation.

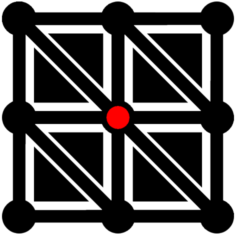
The lattice $\mathcal{S}$ is **not** complemented.

The set $\mathcal{P}(\mathbb{C}_i)$ is a lattice.

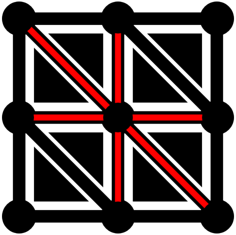
The lattice $\mathcal{P}(\mathbb{C}_i)$ is complemented.

Dimensional operators

$\mathcal{P}(\mathbb{C}_i) \to \mathcal{P}(\mathbb{C}_j)$	$\mathcal{P}(\mathbb{C}_j) \to \mathcal{P}(\mathbb{C}_i)$
$\delta^{+}_{i,j}(X) = \{x \in \mathbb{C}_j \mid \exists y \in X \text{ and } y \subseteq x\}$	



(a) X



(b) $\delta^{+}_{0,1}(X)$

Dimensional operators

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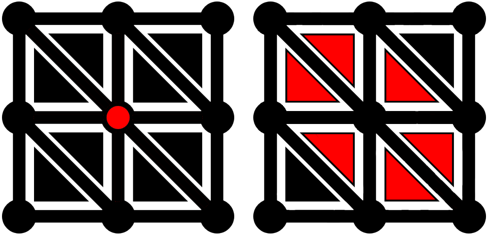

(a) X (b) $\delta^+_{0,2}(X)$

Dimensional operators

$\mathcal{P}(\mathbb{C}_i) \to \mathcal{P}(\mathbb{C}_j)$	$\mathcal{P}(\mathbb{C}_j) \to \mathcal{P}(\mathbb{C}_i)$
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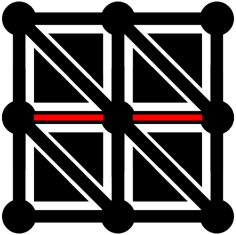
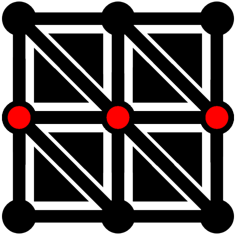


(a) X



(b) $\delta^{-}_{1,0}(X)$

Dimensional operators

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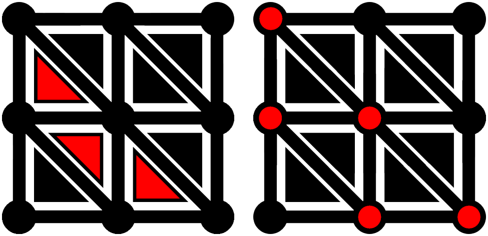
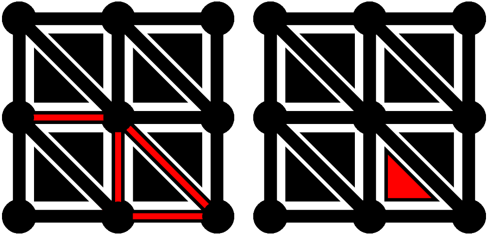


(a) X (b) $\delta^-_{2,0}(X)$

Dimensional operators

$\mathcal{P}(\mathbb{C}_i) \to \mathcal{P}(\mathbb{C}_j)$	$\mathcal{P}(\mathbb{C}_j) \to \mathcal{P}(\mathbb{C}_i)$
$\delta^+_{i,j}(X) = \{x \in \mathbb{C}_j \mid \exists y \in X \text{ and } y \subseteq x\}$	$\delta^-_{j,i}(X) = \{x \in \mathbb{C}_i \mid \exists y \in X \text{ and } x \subseteq y\}$
$\varepsilon^+_{i,j}(X) = \{x \in \mathbb{C}_j \mid \forall y \in \mathbb{C}_i, y \subseteq x \implies y \in X\}$	



(a) X (b) $\varepsilon^+_{1,2}(X)$

Dimensional operators

$\mathcal{P}(\mathbb{C}_i) \to \mathcal{P}(\mathbb{C}_j)$	$\mathcal{P}(\mathbb{C}_j) \to \mathcal{P}(\mathbb{C}_i)$
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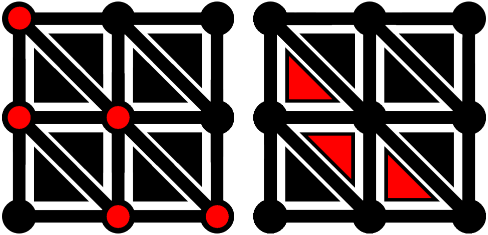

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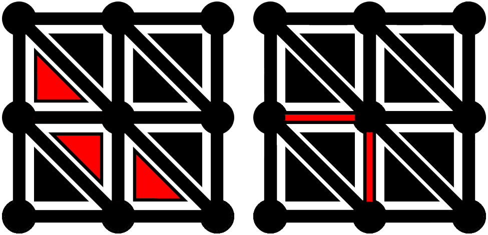

(a) X (b) $\varepsilon^-_{2,1}(X)$

Dimensional operators

$\mathcal{P}(\mathbb{C}_i) \rightarrow \mathcal{P}(\mathbb{C}_j)$	$\mathcal{P}(\mathbb{C}_j) \rightarrow \mathcal{P}(\mathbb{C}_i)$
$\delta^+_{i,j}(X) = \{x \in \mathbb{C}_j \mid \exists y \in X \text{ and } y \subseteq x\}$	$\delta^-_{j,i}(X) = \{x \in \mathbb{C}_i \mid \exists y \in X \text{ and } x \subseteq y\}$
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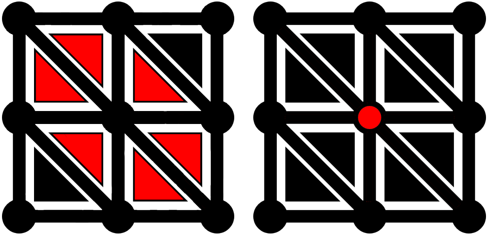
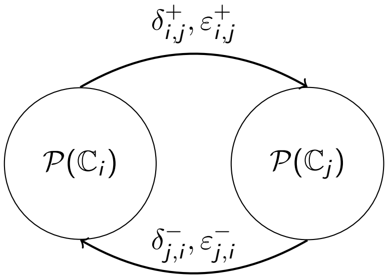


(a) X (b) $\varepsilon^-_{2,0}(X)$

Adjunction, duality

Theorem

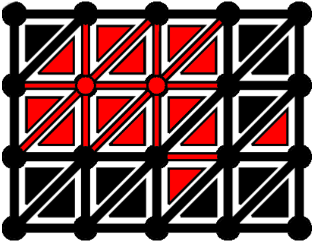
- Pairs $(\varepsilon^-_{j,i}, \delta^+_{i,j})$ and $(\varepsilon^+_{i,j}, \delta^-_{j,i})$ are adjunctions acting between $\mathcal{P}(\mathbb{C}_i)$ and $\mathcal{P}(\mathbb{C}_j)$.
- Operators $\delta^+_{i,j}$ and $\varepsilon^+_{i,j}$ are dual w.r.t. the complement.
- Operators $\delta^-_{j,i}$ and $\varepsilon^-_{j,i}$ are dual w.r.t. the complement.



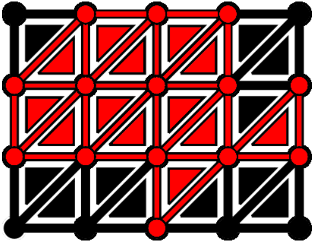
The *closure* and *star* operators

Closure: $\Diamond : \mathcal{S} \to \mathcal{C}$

$$\Diamond(X) = \bigcup_{i \in [0 \ldots n], j \in [0 \ldots i]} \delta^{-}_{i,j}(X) \quad (4)$$



(a) A star Y.

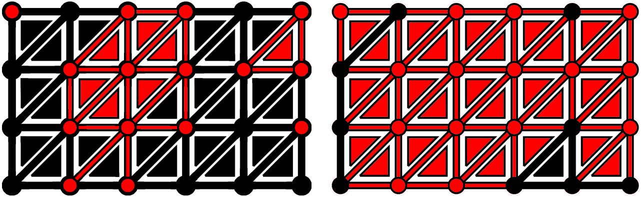


(b) $\Diamond(Y)$.

The *closure* and *star* operators

Star: $\bigstar : \mathcal{C} \rightarrow \mathcal{S}$

$$\bigstar(X) = \bigcup_{i \in [0 \ldots n], j \in [0 \ldots i]} \delta^{+}_{j,i}(X) \quad (4)$$

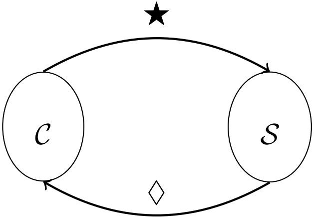


(a) A complex X.   (b) $\bigstar(X)$.

Morphological properties of $\Diamond$ and $\bigstar$:

Dilations:

- Operator $\Diamond$ is a *dilation*, associating elements of $\mathcal{S}$ to $\mathcal{C}$.
- Operator $\bigstar$ is a *dilation*, associating elements of $\mathcal{C}$ to $\mathcal{S}$.



Finding the adjoint erosion

Property:

$$\forall X \in \mathcal{S}, \quad \bigstar^A(X) = \bigcup\{Y \in \mathcal{C} \mid \bigstar(Y) \subseteq X\} \tag{5}$$

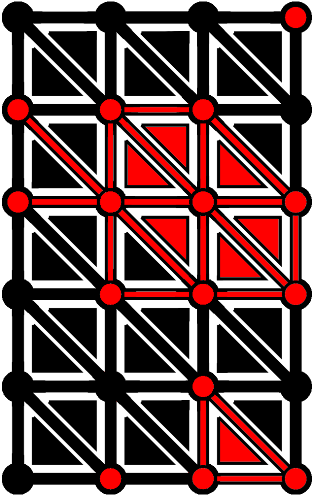
$$\forall X \in \mathcal{C}, \quad \Diamond^A(X) = \bigcup\{Y \in \mathcal{S} \mid \Diamond(Y) \subseteq X\} \tag{6}$$

Duality property:

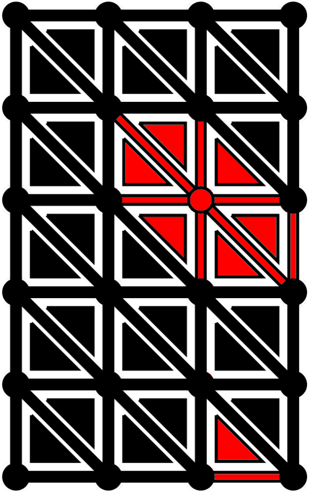
$$\forall X \in \mathcal{S}, \quad \bigstar^A(X) = \overline{\bigstar(\overline{X})} \tag{7}$$

$$\forall X \in \mathcal{C}, \quad \Diamond^A(X) = \overline{\Diamond(\overline{X})} \tag{8}$$

Examples:

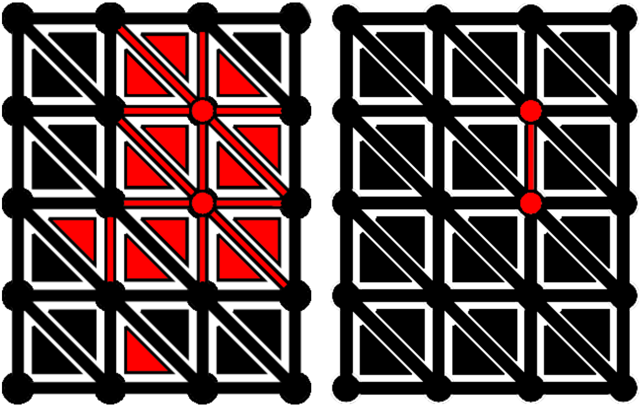


(a) A complex X.



(b) $\Diamond^A(X)$.

Examples:



(c) A star Y.   (d) $\bigstar^A(Y)$.

Adjuntions acting on the same lattice

Operators acting on $\mathcal{C}$:

$$\delta = \Diamond \circ \bigstar \tag{9}$$

$$\varepsilon = \bigstar^{A} \circ \Diamond^{A} \tag{10}$$

The pair (ε, δ) is an adjunction.

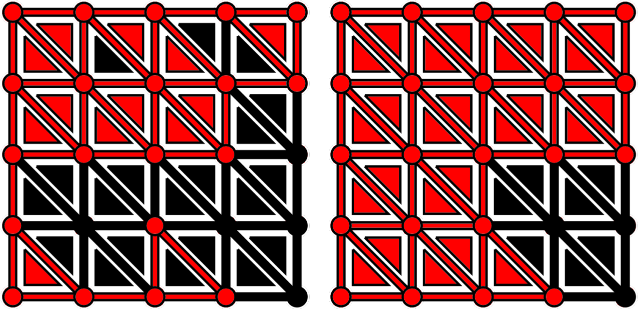
Closing on complexes

$$\phi = \varepsilon \circ \delta \tag{11}$$

Opening on complexes

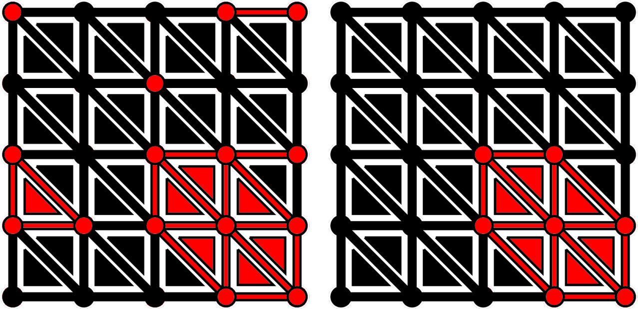
$$\gamma = \delta \circ \varepsilon \tag{12}$$

Examples:



(a) A complex Y. (b) $\phi(Y)$.

Examples:



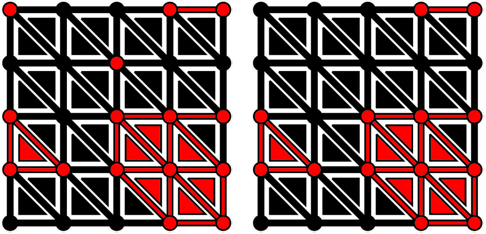
(c) A complex Z. (d) $\gamma(Z)$.

Dimensional filters on complexes

Dimensional opening:

$$\gamma_{d/(n+1)}(X) = \bigcup \left\{ \delta^-_{j,i}(X_j) \mid j \in [d,n], i \in [0,j] \right\} \quad (13)$$

Cells of X with dimension greater than or equal to d.



(a) Z (b) $\gamma_{1/3}(Z)$

Dimensional filters on complexes

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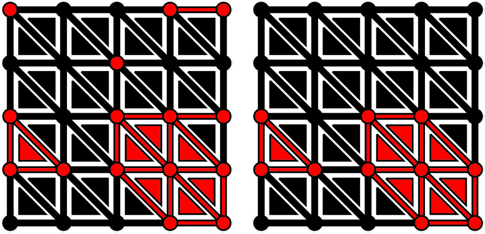


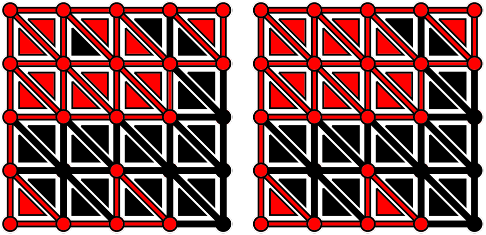
(a) Z (b) $\gamma_{2/3}(Z)$

Dimensional filters on complexes

Dimensional closing:

$$\phi_{d/(n+1)}(X) = X \cup \left(\bigcup_{j \in [n-d,n]} \varepsilon^{+}_{n-d,j}(X_{n-d}) \right) \qquad (14)$$

The set X and the cells of $\mathbb{C}$ whose elements of dimension between 0 and $(n-d)$ belong to X.



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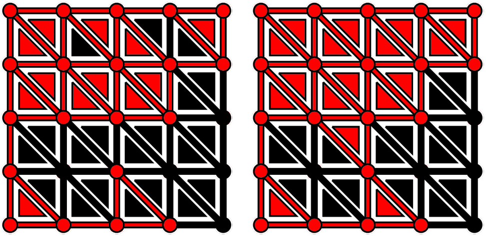


Extending γ and ϕ

We combine the previously defined filters:

$$\Gamma_{k/(n+1)} = \delta^{i} \circ \gamma_{d/(n+1)} \circ \varepsilon^{i} \quad (15)$$

$$\Phi_{k/(n+1)} = \varepsilon^{i} \circ \phi_{d/(n+1)} \circ \delta^{i} \quad (16)$$

where i and d denote respectively the quotient and the remainder of the integer division of k by $(n+1)$

Property:

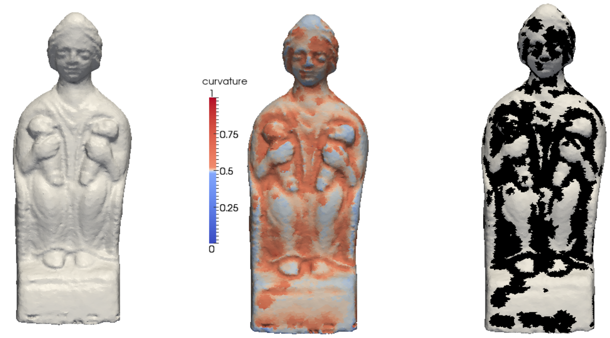
The families $\{\Gamma_{k/(n+1)} \mid k \in \mathbb{N}\}$ and $\{\Phi_{k/(n+1)} \mid k \in \mathbb{N}\}$ are granulometric.

Alternate sequential filters

Definition:

$$ASF_{k/(n+1)} = \begin{cases} \text{identity} & \text{if } k = 0 \\ \Gamma_{k/n+1} \circ \Phi_{k/n+1} \circ ASF_{(k-1)/(n+1)} & \text{otherwise} \end{cases} \quad (17)$$

Illustration on meshes



(a) 3D mesh (b) Curvature map (c) Threshold

Data courtesy of the French Museum Center for Research.

Illustration on meshes

(d) Threshold (e) Dilation (f) Erosion

Illustration on meshes

Original ($ASF_{0/3}$)

Illustration on meshes

$ASF_{1/3}$

Illustration on meshes

$ASF_{2/3}$

Illustration on meshes

$ASF_{3/3}$

Illustration on meshes

$ASF_{4/3}$

Illustration on meshes

$ASF_{5/3}$

Illustration on meshes

$ASF_{6/3}$

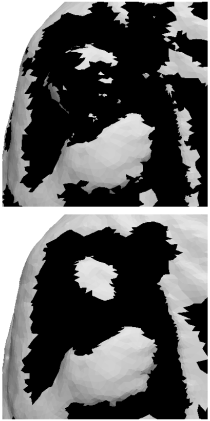
Illustration on meshes

$ASF_{7/3}$

Illustration on meshes

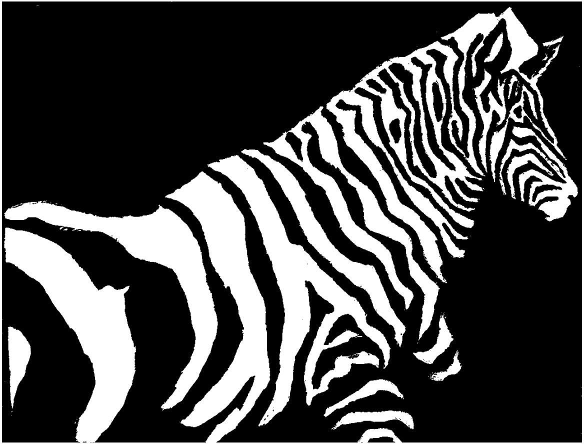
$ASF_{8/3}$

Illustration on meshes



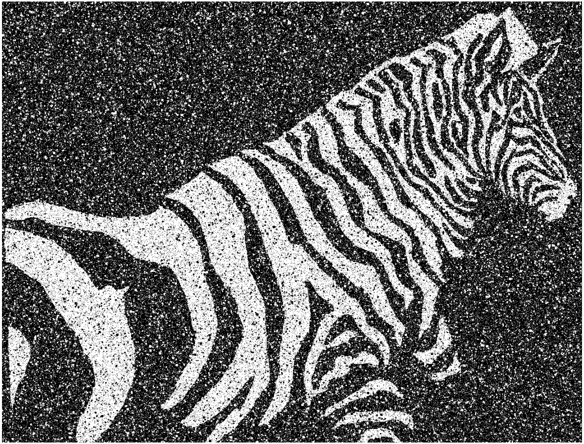
Applicative results

Original



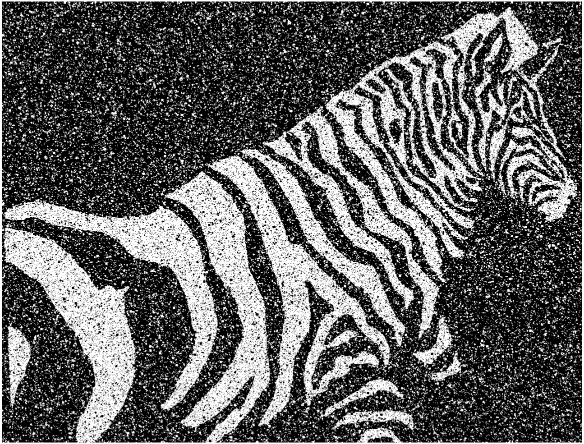
Applicative results

Noisy version ($ASF_{0/3}$)



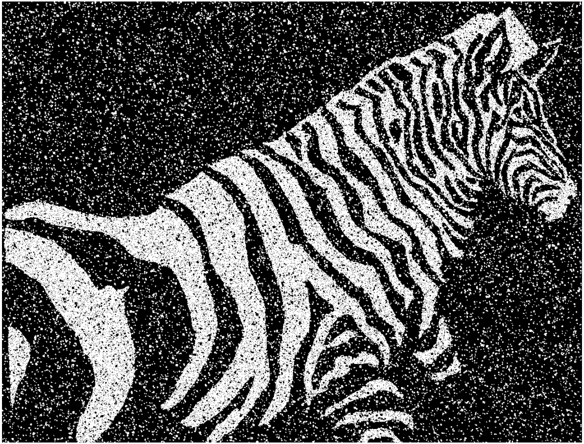
Applicative results

$ASF_{1/3}$



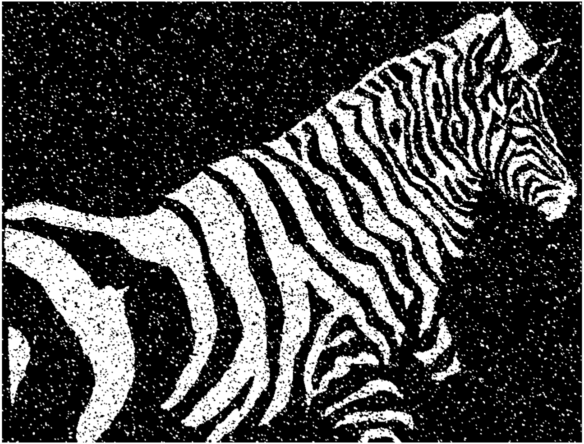
Applicative results

$ASF_{2/3}$



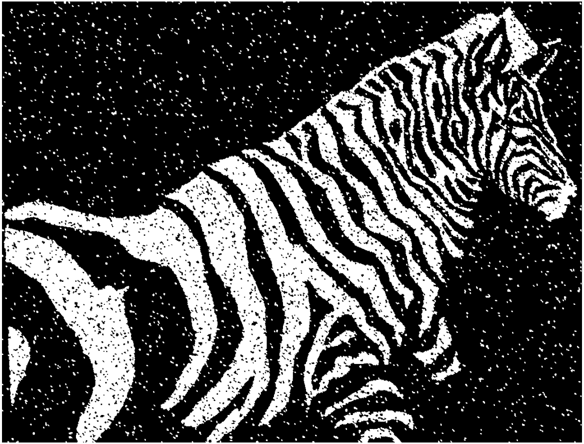
Applicative results

$ASF_{3/3}$



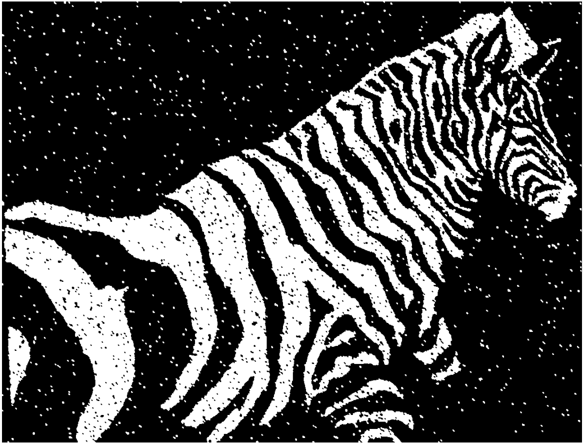
Applicative results

$ASF_{4/3}$



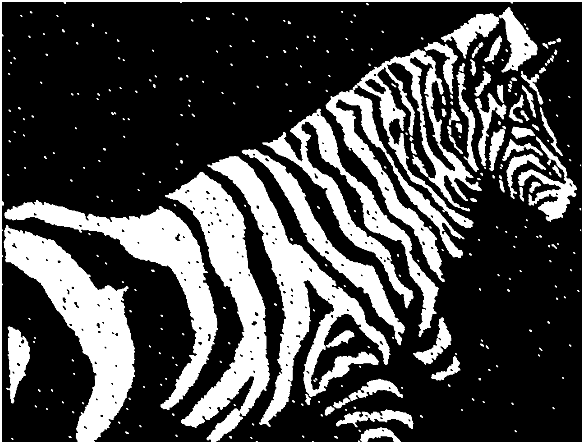
Applicative results

$ASF_{5/3}$



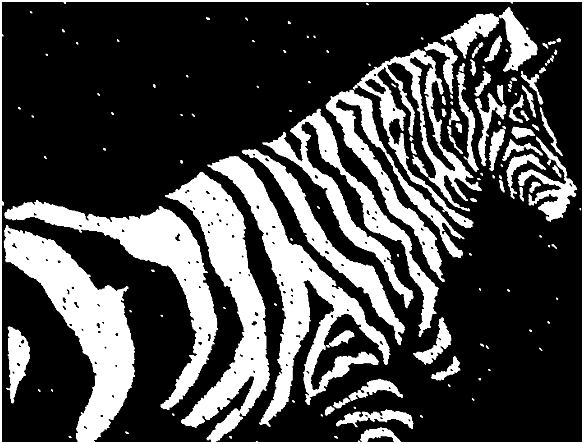
Applicative results

$ASF_{6/3}$



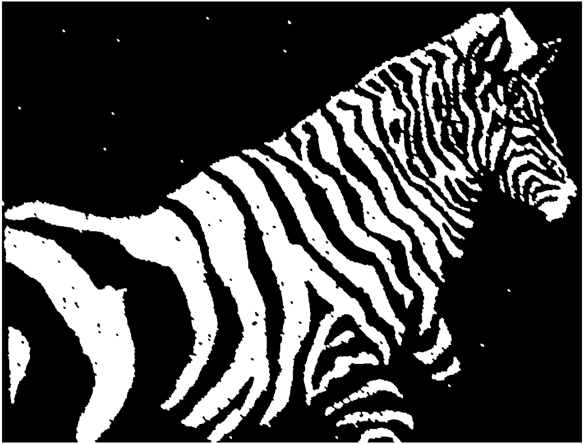
Applicative results

$ASF_{7/3}$



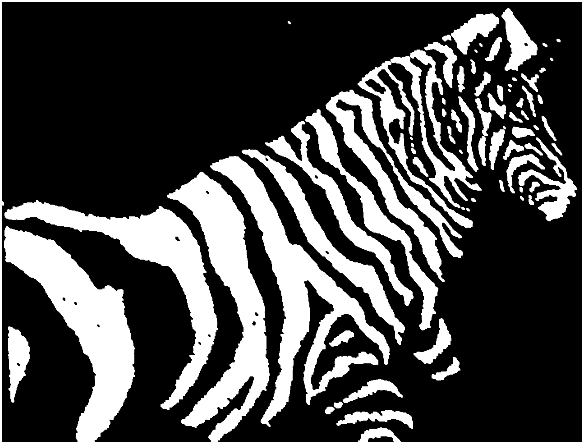
Applicative results

$ASF_{8/3}$

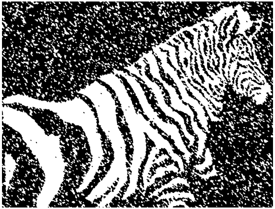


Applicative results

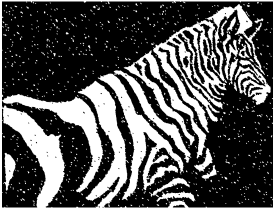
$ASF_{9/3}$



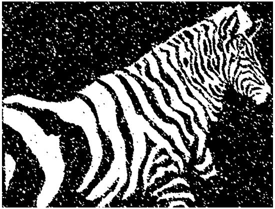
Comparison of results



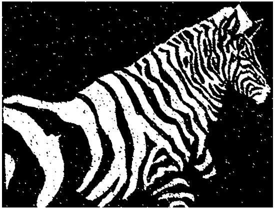
(a) ASF_2. $MSE = 16.14\%$



(b) ASF_6. Triple resolution. $MSE = 4.05\%$



(c) Graph $ASF_{4/2}$. $MSE = 6.88\%$



(d) $ASF_{6/3}$. $MSE = 2.57\%$

Adjuntions acting on the same lattice

Operators acting on $\mathcal{S}$:

$$\Delta = \bigstar \circ \Diamond \tag{18}$$

$$\mathcal{E} = \Diamond^A \circ \bigstar^A \tag{19}$$

Opening $\gamma^s : \mathcal{S} \to \mathcal{S}$

$$\gamma^s = \Delta \circ \mathcal{E}$$

Closing $\phi^s : \mathcal{S} \to \mathcal{S}$

$$\phi^s = \mathcal{E} \circ \Delta$$

Conclusion and future work

It works!

Conclusion and future work

Investigate morphological operators acting on complexes:

- Dilations/Erosions,
- Openings/Closings,
- Granulometries...

To achieve:

- "Smaller resolution".
- Better filtering results on images/meshes.

Conclusion and future work

Future work:

- Investigation of more morphological operators that can be built based on our framework.
- The straightforward extension to weighted simplicial complexes.

Morphological properties of Cl and St:

Dilations:

- Operator Cl is a *dilation*, acting on the lattice $\mathcal{P}(\mathbb{C})$.
- Operator St is a *dilation*, acting on the lattice $\mathcal{P}(\mathbb{C})$.

Property:

$$\forall X \in \mathcal{P}(\mathbb{C}), \quad St^A(X) = \bigcup \{Y \in \mathcal{P}(\mathbb{C}) \mid St(Y) \subseteq X\} \quad (20)$$

$$\forall X \in \mathcal{P}(\mathbb{C}), \quad Cl^A(X) = \bigcup \{Y \in \mathcal{P}(\mathbb{C}) \mid Cl(Y) \subseteq X\} \quad (21)$$

Granulometries based on Cl and St

- **Idempotent**: The presented operators are idempotent.
- **Composing**: The operators St and St^A result *stars*, not complexes. If X is a subcomplex, $Cl(St(X))$ is a subcomplex, but the adjoint operator $St^A(Cl^A(X))$ is a star.

A confusing equivalence

$$\forall X \in \mathcal{P}(\mathbb{C}), \quad St^A(X) = \bigcup \{Y \in \mathcal{P}(\mathbb{C}) \mid St(Y) \subseteq X\}$$

$$\forall X \in \mathcal{P}(\mathbb{C}), \quad Cl^A(X) = \bigcup \{Y \in \mathcal{P}(\mathbb{C}) \mid Cl(Y) \subseteq X\}$$

$$\forall X \in \mathcal{S}, \quad \bigstar^A(X) = \bigcup \{Y \in \mathcal{C} \mid \bigstar(Y) \subseteq X\}$$

$$\forall X \in \mathcal{C}, \quad \Diamond^A(X) = \bigcup \{Y \in \mathcal{S} \mid \Diamond(Y) \subseteq X\}$$

The two following propositions hold true:

$$\forall X \in \mathcal{C}, \Diamond^A(X) = St^A(X); \text{ and} \tag{22}$$

$$\forall Y \in \mathcal{S}, \bigstar^A(Y) = Cl^A(Y). \tag{23}$$