An Improved Riemannian Metric Approximation for Graph Cuts

Ondřej Daněk and Pavel Matula

Centre for Biomedical Image Analysis
Masaryk University, Brno
Czech Republic

DGCI 2011 / Nancy

About Me

- Ph.D student
- Centre for Biomedical Image Analysis
  Faculty of Informatics, Masaryk University
  Brno, Czech Republic
- Contact: http://cbia.fi.muni.cz

Outline

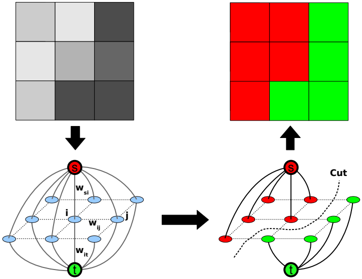
1. Introduction
   - Motivation
   - State of the Art
2. Proposed Method
3. Experimental Results
   - Theoretical tests
   - Practical tests
4. Conclusions

Graph Cut Based Image Segmentation

Reference

Y. Boykov and G. Funka-Lea. *Graph cuts and effcient n-d image segmentation*. International Journal of Compututer Vision, 70(2):109-131, 2006.

- Image segmentation formulated as an discrete energy optimization problem
- Energy function is embedded in a specially designed graph
- Optimal solution obtained by finding a minimum cut



Properties and Challenges

Advantages:

- Global optima
- Polynomial time algorithms
- Straightforward integration of hard constraints
- Applicable in N-D space

Challenges:

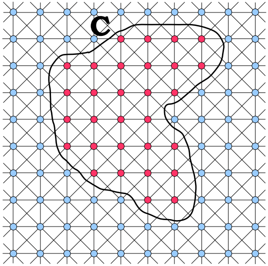
- Optimization of “length” dependent energy terms
- Popular segmentation models:
  - Chan-Vese model - minimizes the intra-region intensity variance and the Euclidean length of the segmentation boundary
  - Geodesic active contours - segmentation boundary defined as a geodesic in an image-based N-D Riemannian space

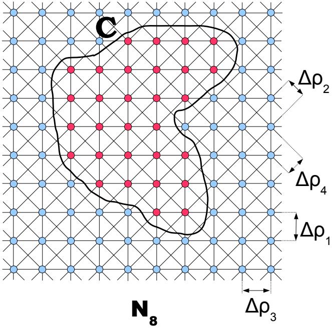
Cut Metrics

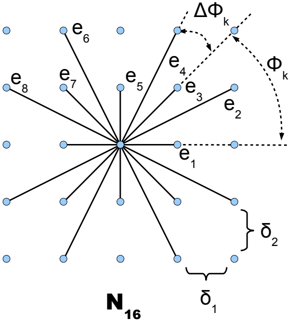
Reference

Y. Boykov and V. Kolmogorov. *Computing geodesics and minimal surfaces via graph cuts*. Proceedings of the Ninth IEEE International Conference on Computer Vision, pp. 26-33, vol. 1, 2003.

- Correspondence of cuts and contours in grid graphs
- Cut cost approximates Euclidean/Riemannian length of a corresponding contour
- Find geodesics and minimal surfaces (satisfying constraints) by finding minimum cuts







$$|C|_{\mathcal{E}} = \frac{1}{2}\int_{\mathcal{L}} n_c(l)\,dl \qquad w_k^{\mathcal{E}} = \frac{\Delta\rho_k \Delta\phi_k}{2} \qquad |\mathcal{C}|_{\mathcal{G}} \xrightarrow[\sup \Delta\phi_k \to 0, \sup ||e_k|| \to 0]{\delta_1, \delta_2 \to 0} |C|_{\mathcal{E}}$$

Riemannian Metrics and Method Issues

Riemannian metrics:

- A smoothly varying metric tensor M defined at each node
- Approximating edge weights (2D):

$$w_k^{\mathcal{R}} = w_k^{\mathcal{E}} \cdot \frac{\det M}{(u_k^T \cdot M \cdot u_k)^{3/2}}$$

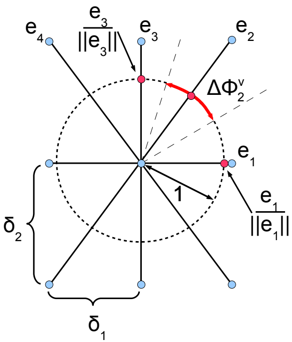
Issues:

- Computation of $\Delta\phi_k$
  - Not invariant to horizontal and vertical mirroring
  - Extension to 3D unclear, no explicit method
- Large error in case of Riemannian metrics for common neighbourhoods

Voronoi Based $\Delta\phi_k$ Partitioning

Reference

O. Daněk and P. Matula. *Graph cuts and approximation of the euclidean metric on anisotropic grids*. VISAPP '10: International Conference on Computer Vision Theory and Applications. vol. 2, pp. 68-73 (2010)



- $\Delta\phi_k^v$ - Measure of lines closest to e_k in terms of their angular orientation
- Computed via Voronoi diagram on a unit hypersphere
- Invariant to mirroring, generalizes to 3D

CENTRE FOR BIOMEDICAL IMAGE ANALYSIS

Decomposition of Constant Metrics

Constant Riemannian metric:

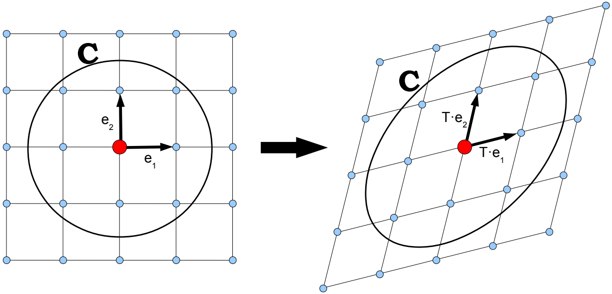
- Non-zero positive definite symmetric matrix M
- $||u||_{\mathcal{R}} = \sqrt{u^T \cdot M \cdot u}$
- Two real-valued eigenvalues λ_1 and λ_2, corresponding to eigenvectors u_1 and u_2
- Represents a space dilation by $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$ in the direction of u_1 and u_2, respectively

Transformation matrix T

- Same eigenvectors as M, but eigenvalues $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$
- $M = T^T \cdot T$

CENTRE FOR BIOMEDICAL IMAGE ANALYSIS

Space Projection Trick

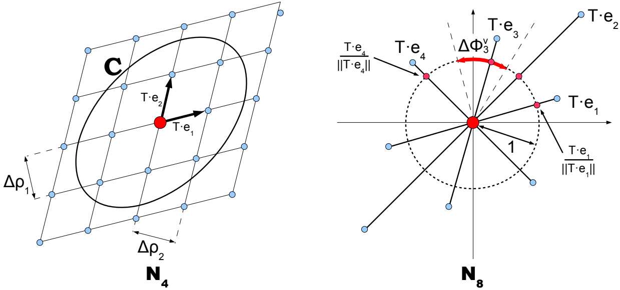


$$
\begin{aligned}
d_{\mathcal{E}}(T \cdot u, T \cdot v) &= ||T \cdot (u - v)||_{\mathcal{E}} \\
&= \sqrt{(T \cdot (u - v))^T \cdot (T \cdot (u - v))} \\
&= \sqrt{(u - v)^T \cdot T^T \cdot T \cdot (u - v)} \\
&= \sqrt{(u - v)^T \cdot M \cdot (u - v)} \\
&= ||u - v||_{\mathcal{R}} \\
&= d_{\mathcal{R}}(u, v)
\end{aligned}
$$

Key Observation

- Projected space is Euclidean
- Transformation is linear $\Rightarrow$ number of intersections with each family of lines is preserved
- Corollary:
  - Cauchy-Crofton formula for Euclidean spaces applies
  - Edge weight formula for Euclidean spaces can be used considering the transformed set of lines
  - Imprecise Riemannian edge weight formula is bypassed

Edge Weights Computation



$$w_k^{\mathcal{R}} = \frac{\Delta\rho_k \Delta\phi_k^v}{2} \qquad \Delta\rho_k = \frac{\det T}{||T \cdot e_k||_{\mathcal{E}}} = \frac{\sqrt{\det M}}{||e_k||_{\mathcal{R}}}$$

CENTRE FOR BIOMEDICAL IMAGE ANALYSIS

Non-constant Metrics and Extension to 3D

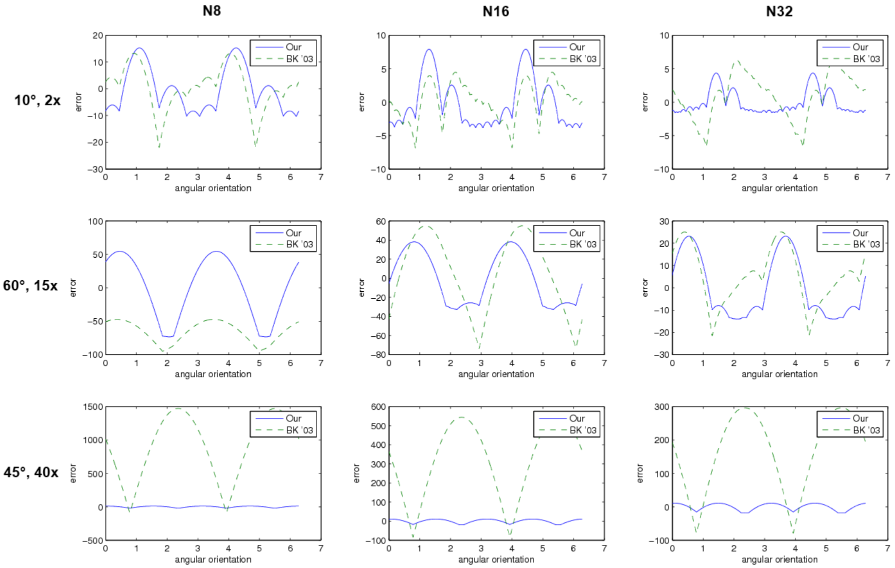
Non-constant metrics:

- Different matrix M is considered in each node to compute the edge weights

Extension to 3D:

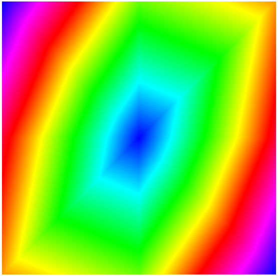
- Derived the same way from Cauchy-Crofton formula for surface area
- $w_k^{\mathcal{R}} = \frac{\Delta\rho_k \Delta\phi_k^v}{\pi}$
- ϕ_k^v is the Voronoi partitioning of a unit sphere surface among the points $\frac{T \cdot e_k}{||T \cdot e_k||}$
- $\Delta\rho_k$ is the line density, the same formula as in 2D applies

Metrication Error for Straight Lines in 2D

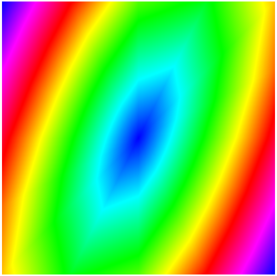


Distance Maps

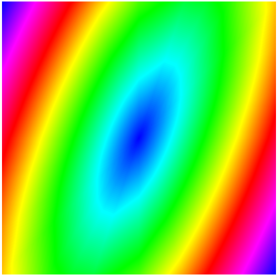
$$M = \begin{pmatrix} 17 & 6 \\ 6 & 5 \end{pmatrix}$$



(a) $\mathcal{N}_8$



(b) $\mathcal{N}_{16}$



(c) $\mathcal{N}_{32}$

Catenoid Reconstruction

(a) $\mathcal{N}_{26}$ (b) $\mathcal{N}_{98}$ (c) Continuous

Image Segmentation - Cell Nuclei

- Image derived anisotropic metric tensor constructed in each point. Minimal separating geodesic is found.

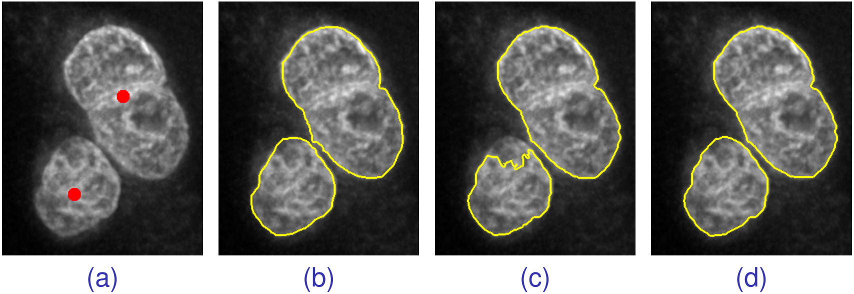
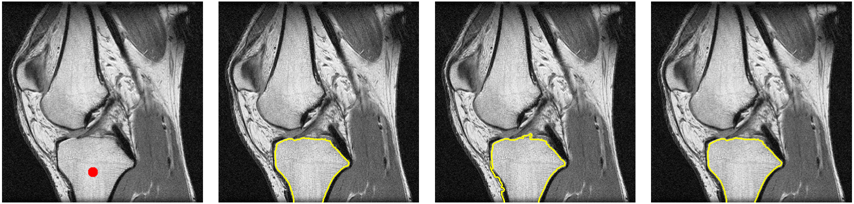


Figure: (a) Image data and foreground seeds. (b) Continuous maximum flow. (c) Combinatorial graph cuts, BK method, $\mathcal{N}_{16}$. (d) Combinatorial graph cuts, proposed method, $\mathcal{N}_{16}$.

Image Segmentation - Knee MRI

- Image derived anisotropic metric tensor constructed in each point. Minimal separating geodesic is found.



(a) (b) (c) (d)

Figure: (a) Image data and foreground seed. (b) Continuous maximum flow. (c) Combinatorial graph cuts, BK method, $\mathcal{N}_{16}$. (d) Combinatorial graph cuts, proposed method, $\mathcal{N}_{16}$.

Image Segmentation - Synthetic Data

- Image derived anisotropic metric tensor constructed in each point. Minimal separating geodesic is found.

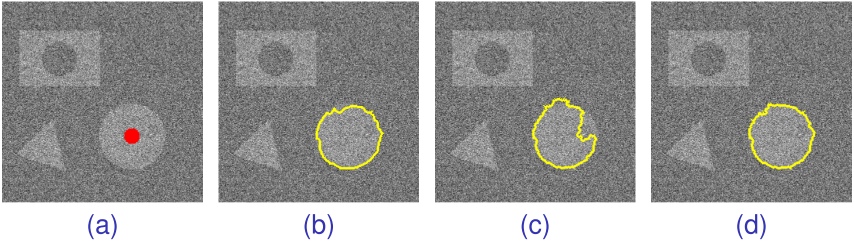


Figure: (a) Image data and foreground seed. (b) Continuous maximum flow. (c) Combinatorial graph cuts, BK method, $\mathcal{N}_{16}$. (d) Combinatorial graph cuts, proposed method, $\mathcal{N}_{16}$.

Conclusions

Alternative method for Riemannian metric approximation via graph cuts presented

- Advantages:
  - Smaller error than existing approaches
  - Explicit formula for both 2D and 3D
  - Straightforward integration into existing algorithms for improved precision
- Disadvantages:
  - Computation of spherical Voronoi diagram in 3D is slow
    - Can be computed only once for scalar (isotropic) or constant metrics
  - Still less precise than continuous methods

Future work:

- Better/faster approximation for discrete graph cuts?

CBIA CENTRE FOR BIOMEDICAL IMAGE ANALYSIS

End of the Talk

Thank you for your attention!

xdanek2@fi.muni.cz

http://cbia.fi.muni.cz
http://cbia.fi.muni.cz/projects/graph-cut-library.html

CENTRE FOR BIOMEDICAL IMAGE ANALYSIS