# Characterization of {-1,0,1} valued functions in DT under sets of four directions

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# Discrete Tomography (DT)

Herman, Kuba: *Discrete Tomography: Foundations, Algorithms, and Applications*, Birhauser 1999

"We assume there is a domain, which may itself be discrete (such as a set of ordered pairs of integers) or continuous (such as Euclidean space). We further assume that there is an unknown function ***f*** whose range is known to be a given discrete set (usually of real numbers). The problems of DT, as we perceive the field, have to do with determining ***f*** (perhaps only partially, perhaps only approximately) from weighted sums over subsets of its domain in the discrete case and from weighted integrals over subspaces of its domain in the continuous case"

Usually : $A = \{(i, j) \in Z^2 : 0 \le i < m, 0 \le j < n\}$ $\quad f : A \to \{0,1\}$

# Reconstruction problem

a direction $(a, b)$

$gcd(a,b)=1\,;a\geq 0\,;b=1, if\ a=0$

$ay=bx+t\,, t\in Z$

line sums in direction $(a, b)$ : $\sum_{aj=bi+t} f(i,j)$

Let A be a grid of size *mxn*, and $S=\{\,(a_d, b_d)\}_{d=1,\dots,k}$.

Suppose $f: A\rightarrow \{0,1\}$ is unknown, but all the line sums are given for $d=1,\dots,k$. Construct a function $g$ : A$\rightarrow$ $\{0,1\}$ s.t.

$$\sum_{a_d j=b_d i+t} f(i,j)=\sum_{a_d j=b_d i+t} g(i,j) \quad for\ d=1,\dots,k\,, t\in Z$$

# Some questions

Is the reconstructed function unique?

Is there another function having the same line sums as the reconstructed one?

Approaches and results:

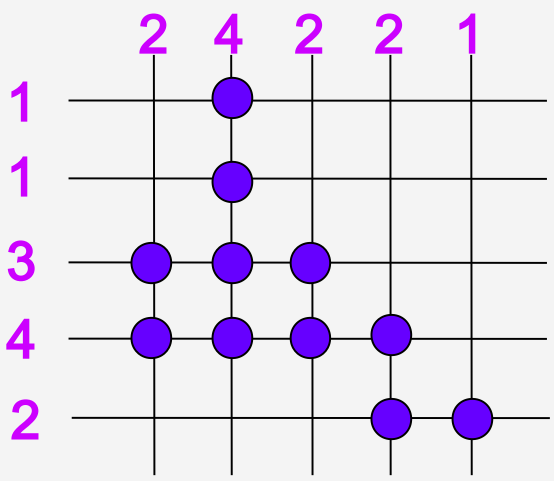
- Two directions [Chang],[Ryser]
- Additivity [Fishburn et Al],[Vallejo]
- Geometrical approach [Gardner&Gritzmann],[Daurat], [BDNP]
- Algebraic approach [Hadju&Tijdman]

# Algebraic approach

$g : A \to Z$

Generating function of $g$: $G_g(x, y) = \sum g(i, j)\, x^i y^j$

$$G_g(x, y) = x y^4 + x^4 + x^3 y + x^2 y^2 + x y^3 + x^3 + x^2 y + x y + y^2 + y$$




line sums in direction $(a, b)$

are given by the remainder of $G_g(x, y)$

divided by: $x^a y^b - 1, if\ a \geq 0, b > 0$

$x^a - y^{-b}, if\ a > 0, b \leq 0$

## If $(a,b)$ is a direction:

$$f_{(a,b)}(x,y): \quad \begin{array}{l} x^a y^b - 1, if\ a>0, b>0 \\ x^a - y^{-b}, if\ a>0, b<0 \\ x-1, if\ a=1, b=0 \\ y-1, if\ a=0, b=1 \end{array}$$

$$S=\{(a_i,b_i)\}_{i=1,\ldots,k} \qquad F_S(x,y)=\Pi f_{(a_i,b_i)}(x,y)$$

[HT] *Let g* :A→ Z *having zero line sums along S.*
*Then* $F_S(x,y)$ *divides* $G_g(x,y)$ over Z.

# Known results

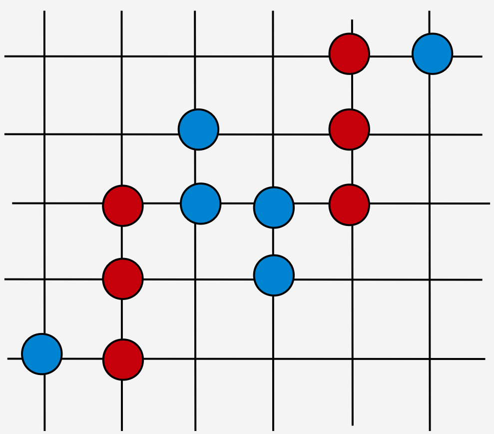
- $S=\{(a_i,b_i)\}_{i=1,\dots,k}$ *is valid for A if* $\sum a_i < m, \sum abs(b_i) < n$
- There exists a valid set *S* for A consisting of *k* directions depending on the size of A s.t. if $g$ :A$\rightarrow$ Z has zero line sums along *S* and $|g| \leq 1$, then $g \equiv 0$
- There exists a valid set *S* for A consisting of *k* directions depending on the size of A s.t. if $g,h$ :A$\rightarrow$ $\{0,1\}$ are tomographically equivalent, then $g \equiv h$ (i.e. S detemines uniquely any subset of A)
- Less than four directions are never sufficient to distinguish subsets of A

# New results (Outline)

- Focus on four directions
- Necessary condition for $S$ to be a set of uniqueness
- Characterization for $G_g(x,y)$ when $|g| \leq 1$ has zero line sums along $S$
- Uniqueness result
- not depending on the grid A

# Necessary condition

Lemma: Let $S$ be any valid set and let $F_S(x,y)=\sum f(i,j)\, x^i y^j$ be the polynomial associated to $S$. If $f(i,j)$ is in {-1,0,1}, then there exist two tomographically equivalent lattice sets with respect to $S$.



$$S=\{(1,0),(1,2),(1,1),(2,1)\}$$

$$F_S(x,y)=(x-1)(x y^2-1)(x y-1)(x^2 y-1)$$

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Lemma [H]: Let $S=\{(a_i, b_i)\}_{i=1,\dots,k}$ be any valid set.

Suppose $F_S(x,y)=\sum f(i,j)\, x^i y^j$ has some coefficient $f(i,j)$ outside $\{-1,0,1\}$. Then there exist two disjoint subsets $S_1$ and $S_2$ of $S$ s. t. $|S_1| = |S_2| \, mod2$ and

$$\sum_{(a,b)\in S_1}(a,b)=\sum_{(a,b)\in S_2}(a,b)$$

If $k=4$: $S=\{u_1, u_2, u_3, u_4\}$

$F_S(x,y)=\sum f(i,j)\, x^i y^j$ has some coefficient $f(i,j)$ outside $\{-1,0,1\}$

***iff*** there exist two disjoint subsets $S_1$ and $S_2$ *of S s. t.*

$|S_1| = |S_2| \, mod2$ and

$$\boxed{\sum_{(a,b)\in S_1}(a,b)=\sum_{(a,b)\in S_2}(a,b)}$$

Proof:

***if*** *two disjoint subsets* $S_1$ *and* $S_2$ *of* $S$ *exist s. t.* $|S_1| = |S_2| \bmod 2$ *and* $\sum_{(a,b)\in S_1}(a,b) = \sum_{(a,b)\in S_2}(a,b)$, *then* $F_S(x,y)$ *has some coefficient* $f(i,j)$ *outside* $\{-1,0,1\}$

The condition corresponds to:

$$u_1 + u_2 + u_3 = u_4$$

$$u_1 + u_2 = u_3 + u_4$$

Consider the case: $u_1+u_2+u_3=u_4$ where
$u_1=(a,p), u_2=(b,q), u_3=(c,r), u_4=(d,s)$

w.l.o.g. assume:

$$0\leq a\leq b\leq c, b, c\neq 0$$

the coeff. 2 cannot be delated by any other, (by inspection)

Table 1. Monomials of $F_S(x,y)$ in CASE 1

| sign $+$ | sign $-$ |
|---|---|
| $x^{2(a+b+c)}y^{2(p+q+r)}$ | $x^{2a+2b+c}y^{2p+2q+r}$ |
| $x^{2a+b+c}y^{2p+q+r}$ | $x^{2a+b+2c}y^{2p+q+2r}$ |
| $x^{a+2b+c}y^{p+2q+r}$ | $x^{a+2b+2c}y^{p+2q+2r}$ |
| $x^{a+b+2c}y^{p+q+2r}$ | $\boxed{2x^{a+b+c}y^{p+q+r}}$ |
| $x^{a+b}y^{p+q}$ | $x^{a}y^{p}$ |
| $x^{a+c}y^{p+r}$ | $x^{b}y^{q}$ |
| $x^{b+c}y^{q+r}$ | $x^{c}y^{r}$ |
| $1$ | |

## Theorem

If $S$ uniquely determines lattice sets in a finite grid, it must be of the form

$$\boxed{u_1+u_2\pm u_3=u_4}$$

Remark: if $S$ is as before, $S = S_1 \cup S_2$ , where:

- $S_1 =\{u_1, u_2, u_3\}$ and $S_2 =\{ u_4\}$ , if $u_1+u_2 +u_3= u_4$
- $S_1 =\{u_1, u_2 \}$ and $S_2 =\{u_3, u_4\}$ , if $u_1+u_2 =u_3+u_4$

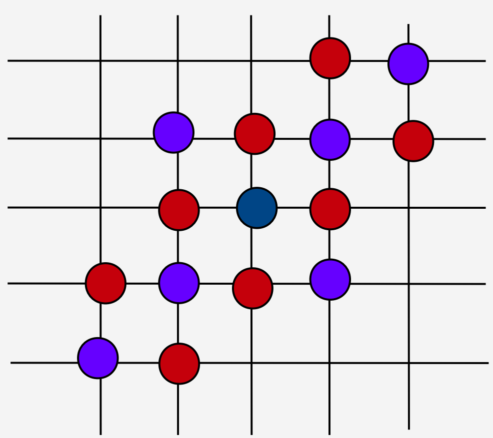
- $S_1 - S_2 =\{ \pm(u_i-u_j), u_i\in S_1, u_j\in S_2, \}$

# A characterization

$$(1,0)+(1,2)=(0,1)+(2,1)$$

$$F_S(x,y)=(x-1)(x\,y^2-1)(y-1)(x^2\,y-1)$$

$Q(S)=\{\quad\}$ $\qquad$ $P=\{\quad\}$ $\qquad$ $N=\{\quad\}$



$S_1=\{(1,0),(1,2)\}$ $\quad$ $S_2=\{(0,1),(2,1)\}$

$S_1 - S_2 = \pm\{(1,-1),(1,1)\}$

$$P\text{-}P=\{\pm(x-y),\ x\in P,\ y\in P\}$$

$$=\{(0,0),(1,1),(2,2),(3,3),(4,4),(1,-1),(1,4),(4,1),(2,0),(0,2)\}$$

$$N\text{-}N=\{\pm(x-y),\ x\in N,\ y\in N\}$$

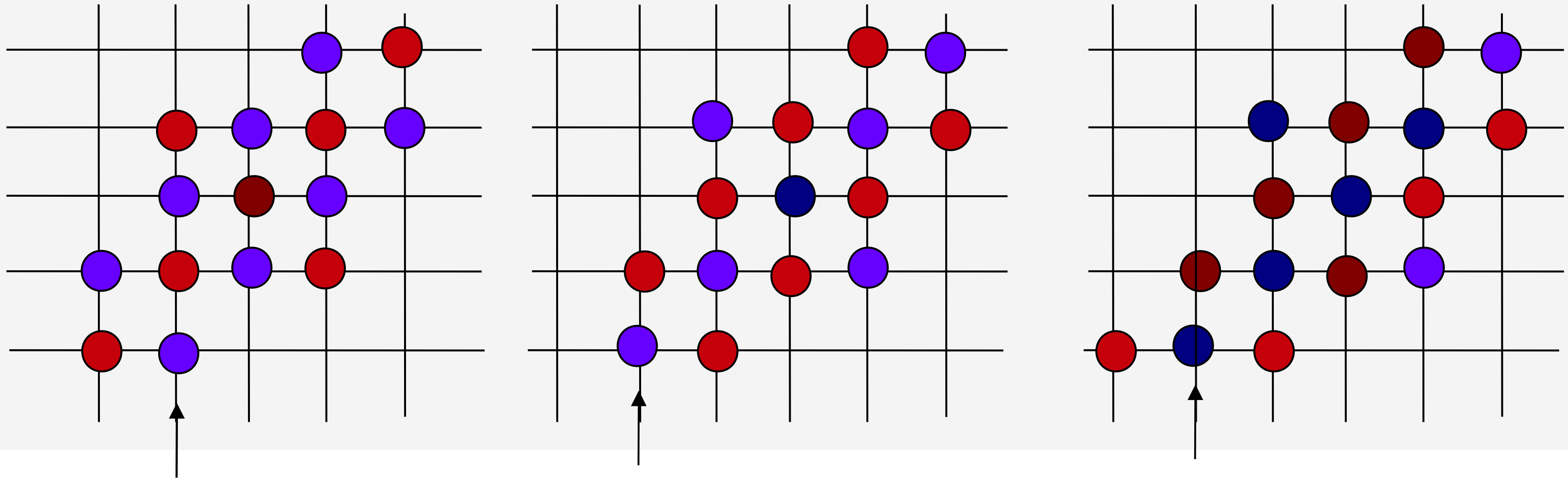
$$P\text{-}N=\{\pm(x-y),\ x\in P,\ y\in N\}$$

$u=(h,k)$

<u>Lemma</u>: If $u$ or $-u$ belongs to $S$, then $(x^h y^k - 1) F_S(x, y)$ has some coefficient greater than 1.

Let $u=(h,k)=(1,0)$

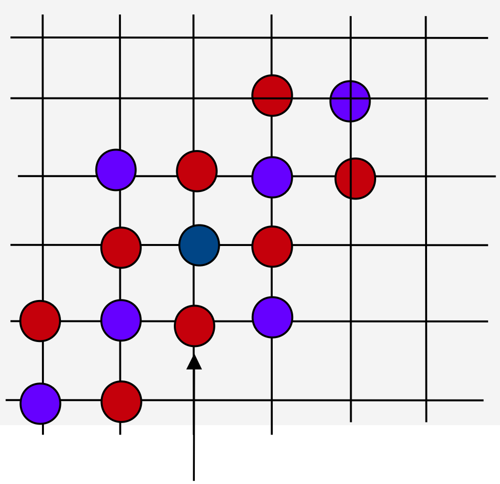
$-F_S(x, y)$ $\quad$ $x F_S(x, y)$ $\quad$ $(x-1) F_S(x, y)$



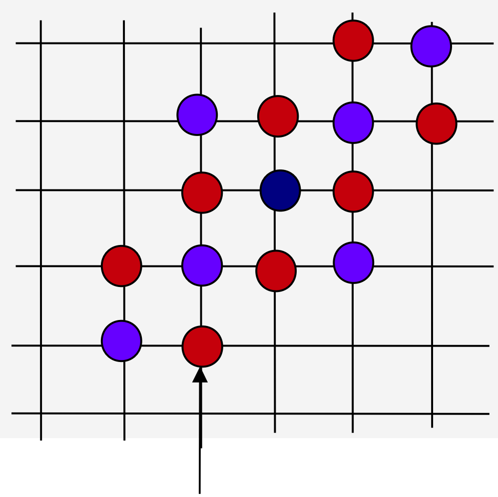
Lemma: for each $u_i = (h_i, k_i)$ in $S$, there exists $u = (h, k)$ in $S_1 - S_2$ such that the coefficient of $x^{h_i} y^{k_i}$ in $(x^h y^k + 1) F_S(x, y)$ is greater than 1.

Let $u = (h,k) = (1,1) = (2,1) - (1,0)$

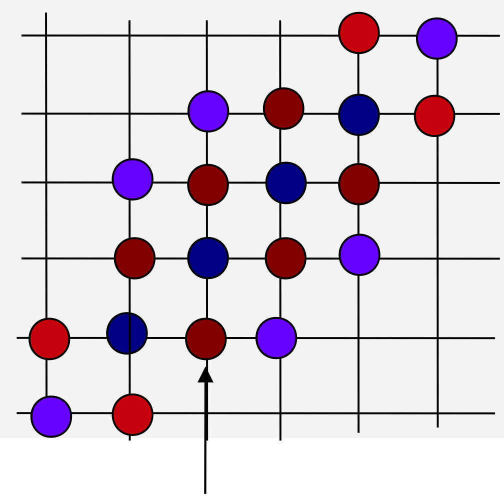
$F_S(x, y)$



$xy\, F_S(x, y)$



$(xy + 1) F_S(x, y)$



**Lemma:** the following hold:

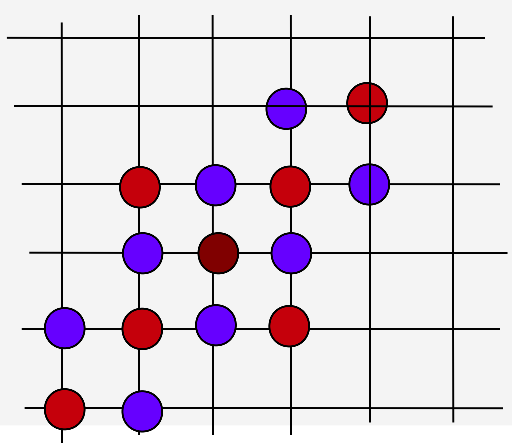
$(x^h y^k - 1) F_S(x,y)$ has coef.s in {-1,0,1} iff $u$=0, or $u$ in $S_1$- $S_2$ and

$u_1+u_2+u_3= u_4$, or $u$ in $(S_1- S_2)$U $\pm(u_1+u_2)$ and $u_1+u_2-u_3= u_4$

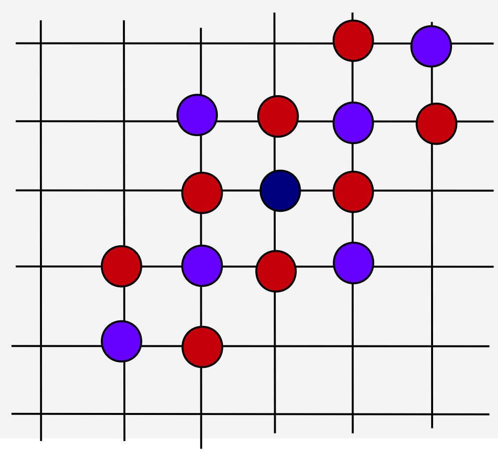
$(x^h y^k + 1) F_S(x,y)$ has coef.s in {-1,0,1} iff $u$ in $S$, or $-u$ in $S$

Example 1:

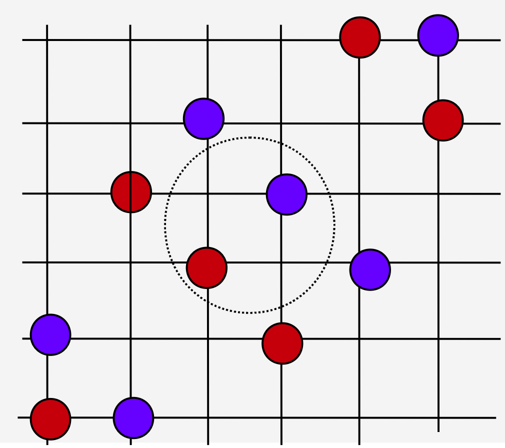
$-F_S(x,y)$



$xy\, F_S(x,y)$



$(xy-1) F_S(x,y)$



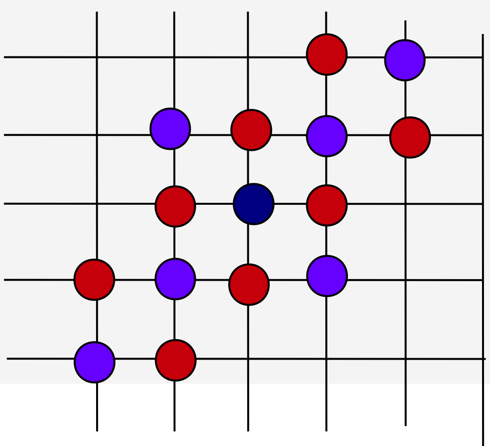
Lemma: the following hold:

$(x^h y^k - 1)\,F_S(x,y)$ has coef.s in {-1,0,1} iff $u$=0, or $u$ in $S_1$- $S_2$ and $u_1$+$u_2$ +$u_3$= $u_4$, or $u$ in ($S_1$- $S_2$)U ±¿($u_1$+$u_2$ ) and $u_1$+$u_2$ -$u_3$= $u_4$

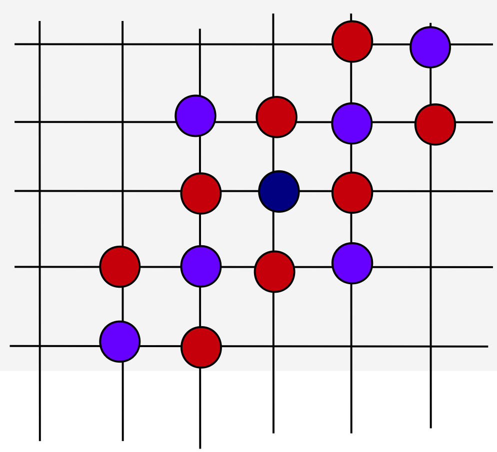
$(x^h y^k + 1)\,F_S(x,y)$ has coef.s in {-1,0,1} iff $u$ in $S$, or -$u$ in $S$

Example 2:

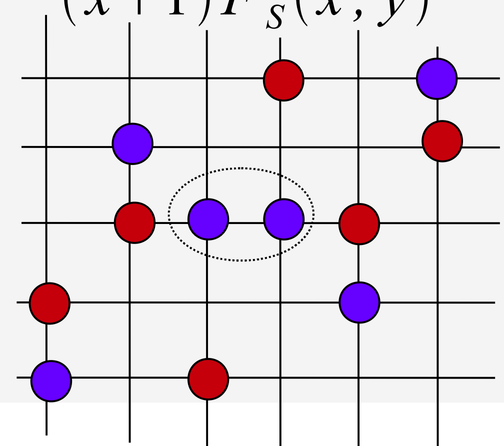
$F_S(x,y)$



$x\,F_S(x,y)$



$(x+1)\,F_S(x,y)$

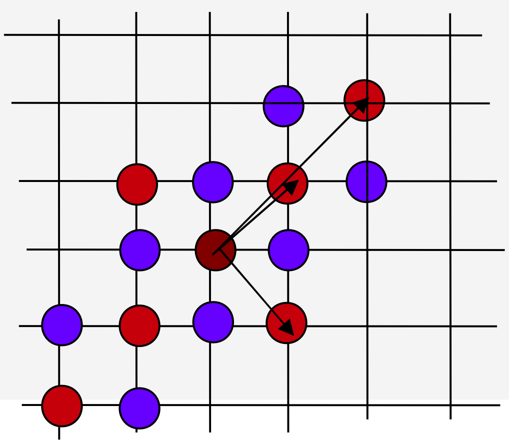


# proof:

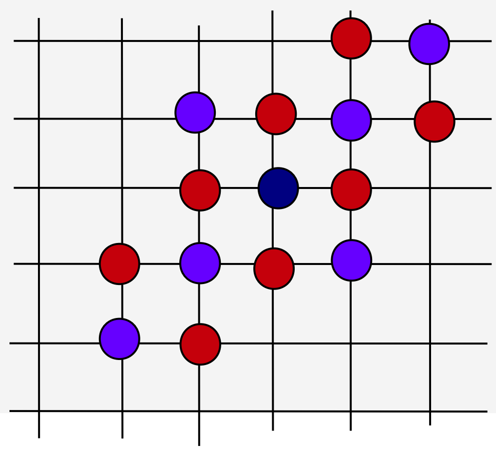
Consider $w = u_1 + u_2 = u_3 + u_4$ and $w + u = z$. *If* $z$ is not in $Q(S)$, $z$ is multiple. So $z$ should be in $Q(S)$.

*If* $u$ *or* $-u$ belongs to $S$, then $(x^h y^k - 1) F_S(x, y)$ has some coefficient greater than 1. Take $u$ in $(S_1 - S_2) \cup \pm(u_1 + u_2)$.

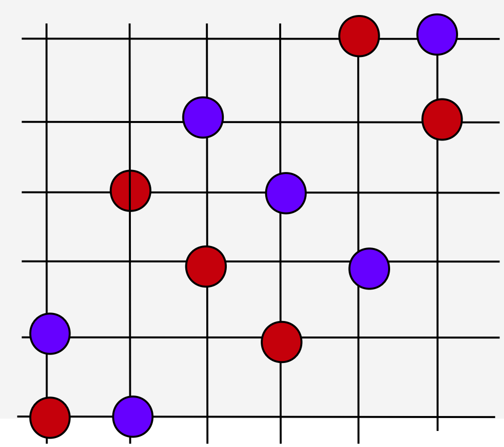
$-F_S(x, y)$



$xy\, F_S(x, y)$



$(xy - 1) F_S(x, y)$

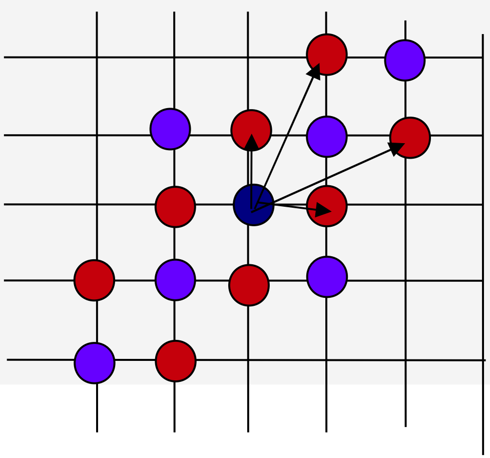


# proof:

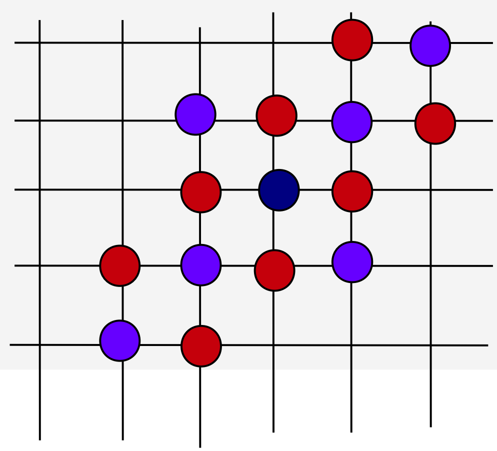
Consider $w = u_1+u_2 = u_3+u_4$ and $w+u=z$. *If* $z$ is not in $Q(S)$, $z$ is multiple. So $z$ should be in $Q(S)$.

*If* $u$ *or* $-u$ belongs to $(S_1 - S_2) \cup \pm(u_1+u_2)$, then $(x^h y^k + 1) F_S(x, y)$ has some coefficient greater than 1. Take $u$ in $S$.

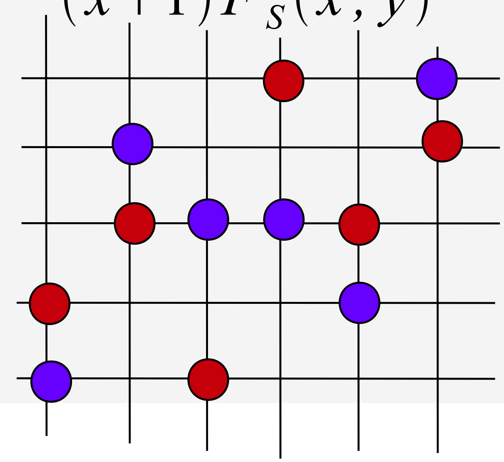
$F_S(x, y)$



$x F_S(x, y)$



$(x+1) F_S(x, y)$



**Theorem 3.** *Let $S = \{u_1, u_2, u_3, u_1 + u_2 \pm u_3\}$ be a set of four lattice directions. Let $g : \mathbb{Z}^2 \to \mathbb{Z}$ be a non trivial function which has zero line sums along the lines corresponding to the directions in $S$, and vanishes outside a finite lattice set. If $|g| \leq 1$ then there exists $r \in \mathbb{N}$ such that*

$$G_g(x, y) = \sum_{t=1}^{r} \delta(t) x^{i(t)} y^{j(t)} F_S(x, y), \tag{8}$$

*where $\delta(t) = \pm 1$, and for each $t \in \{1, ..., r\}$, there exists $t' \in \{1, ..., r\}$ such that the vector $u(t) = (i(t), j(t)) - (i(t'), j(t'))$ satisfies the following conditions*

1. *$u(t) = 0$, or $u(t) \in (S_1 - S_2)$ and $u_4 = u_1 + u_2 + u_3$, or $u(t) \in (S_1 - S_2) \cup \{\pm(u_1 + u_2)\}$ and $u_4 = u_1 + u_2 - u_3$, if $\delta(t) \neq \delta(t')$.*
2. *$u(t) \in S$ or $-u(t) \in S$ if $\delta(t) = \delta(t')$.*

# proof:

By [HT], since $g$ :A→ Z *has zero line sums along S,*

$G_g(x,y) = P(x,y)\, F_S(x,y)$

Any monomial of $P(x,y)$ with |coef.|>1 can be rewrited as sum of monomial with |coef.|=1

So :

Since the coef.s of $G_g(x,y)$ are in {-1,1},

for all $t$, there exists $z_t$ in $Q(S)$ such that $z_t+(i(t'),j(t'))=w+(i(t),j(t))$

$$(\delta(t)x^{i(t)}y^{j(t)}+\delta(t')x^{i(t')}y^{j(t')})F_S(x,y)$$

$$x^{i(t')}y^{j(t')}(\delta(t)x^{i(t)-i(t')}y^{j(t)-j(t')}+\delta(t'))F_S(x,y)$$

Apply the Lemma with $u(t)=(i(t)-i(t'),j(t)-j(t'))$

# Uniqueness result

**Corollary 1.** *$S = \{u_1, u_2, u_3, u_1 + u_2 \pm u_3\}$ be a set of four lattice directions. Let $h, g : \mathbb{Z}^2 \to \{0, 1\}$ be tomographically equivalent with respect to $S$. If*

$$G_{h-g}(x, y) = \sum_{t=1}^{r} \delta(t) x^{i(t)} y^{j(t)} F_S(x, y),$$

*with $\delta(t) = \pm 1$, and the degrees $i(t), j(t)$ do not satisfy the conditions of Theorem 3, then $h = g$.*

# Other results

- *Is it possible to construct Gg(x,y) as in the Theorem?*
- $Gg^h(x,y)=P^h(x,y)F_S(x,y)$
- How to choose a monomial to add to $P^h(x,y)$ in order to decrease the number of |coeff.s|>1 in $Gg^{h+1}(x,y)$
- *Uniqueness in A for valid sets of four directions*
- There exist whole famileis of suitable valid directions

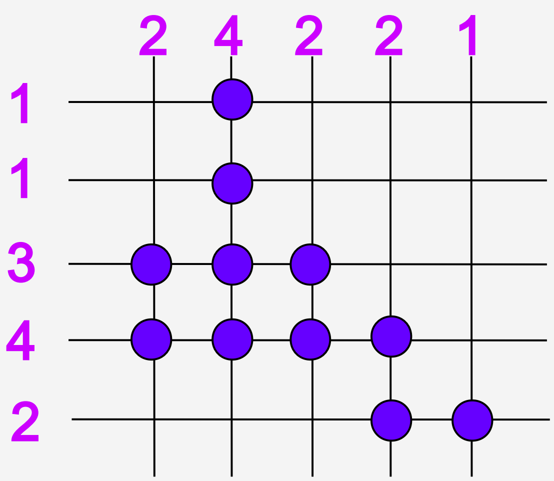
The end

# The setting

$$A=\{(i,j)\in Z^2 : 0\le i<m,\ 0\le j<n\}$$

$F$ subset of A

characteristic function of $F$: $f : A \to \{0,1\}$




a direction $(a,b)$

$gcd(a,b)=1\,; a\ge 0\,; b=1, if\ a=0$

$ay=bx+t\,,\ t\in Z$

line sums in direction $(a,b)$

$$\sum_{aj=bi+t} f(i,j)$$