Mathematical Morphology on Hypergraphs: Preliminary Definitions and Results

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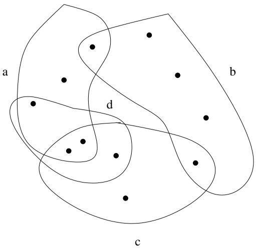
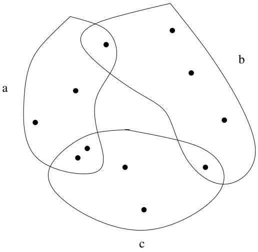
Motivation





- Dilation / erosion of an hypergraph?
- Are these two hypergraphs similar?
- ...

Hypergraphs

- Underlying space: $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V}$ the set of vertices and $\mathcal{E}$ the set of hyperedges.
- Hypergraph: $H = (V, E)$ with $V \subseteq \mathcal{V}$ and $E \subseteq \mathcal{E}$, $E = ((e_i)_{i \in I})$ $(e_i \subseteq V)$.
- Dual hypergraph: $H^* = (V^* \simeq E,\ E^* \simeq (H(x))_{x \in V})$ with $H(x) = \text{star} = \{e \mid x \in e\}$.

Partial ordering and lattice

- On vertices: $(\mathcal{P}(\mathcal{V}), \subseteq)$.
- On hyperedges: $(\mathcal{P}(\mathcal{E}), \subseteq)$.
- On $\mathcal{H}$: $\mathcal{T}$ + partial ordering

$$H = (V, E) \in \mathcal{T} \Leftrightarrow \begin{cases} V \subseteq \mathcal{V} \\ E \subseteq \mathcal{E} \\ \{x \in \mathcal{V} \mid \exists e \in E, x \in e\} \subseteq V \end{cases}$$

Inclusion-based ordering:

$$\forall (H_1, H_2) \in \mathcal{T}^2, H_1 = (V_1, E_1), H_2 = (V_2, E_2),\ H_1 \preceq H_2 \Leftrightarrow \begin{cases} V_1 \subseteq V_2 \\ E_2 \subseteq E_2 \end{cases}$$

$(\mathcal{T}, \preceq)$ = complete lattice
$\bigwedge_i H_i = (\bigcap_i V_i, \bigcap_i E_i)$, $\bigvee_i H_i = (\bigcup_i V_i, \bigcup_i E_i)$
Smallest element: $H_\emptyset = (\emptyset, \emptyset)$, largest element: $\mathcal{H} = (\mathcal{V}, \mathcal{E})$.

Ordering from induced sub-hypergraph:

$$\forall (H_1, H_2) \in \mathcal{T}^2, H_1 \preceq_i H_2 \Leftrightarrow \begin{cases} V_1 \subseteq V_2 \\ E_1 = \{V(e) \cap V_1 \mid e \in E_2\} \end{cases}$$

i.e. H_1 is the sub-hypergraph induced by H_2 for V_1.

$$\forall (H_1, H_2) \in \mathcal{T}^2, H_1 \preceq'_i H_2 \Leftrightarrow \begin{cases} V_1 \subseteq V_2 \\ E_1 \subseteq \{V(e) \cap V_1 \mid e \in E_2\} \end{cases}$$

$(\mathcal{T}, \preceq'_i)$ = complete lattice.
$H_1 \wedge'_i H_2 = (V_1 \cap V_2, \{V(e_1) \cap V_2, V(e_2) \cap V_1 \mid e_1 \in E_1, e_2 \in E_2\} \cap \mathcal{E}\})$,
$H_1 \vee'_i H_2 = (V_1 \cup V_2, E_1 \cup E_2)$.
Smallest element: $H_\emptyset = (\emptyset, \emptyset)$, largest element: $\mathcal{H} = (\mathcal{V}, \mathcal{E})$.

Ordering from partial hypergraph:

$$H_1 \preceq_p H_2 \Leftrightarrow \begin{cases} V_1 = V_2 \\ E_1 \subseteq E_2 \end{cases}$$

Ordering from sub-hypergraph:

$$H_1 \preceq_s (\preceq'_s) H_2 \Leftrightarrow \begin{cases} V_1 \subseteq V_2 \\ E_1 = (\subseteq)\{e \mid e \in E_2 \text{ and } V(e) \subseteq V_1\} \end{cases}$$

Algebraic dilation and erosion

- $(\mathcal{T}, \preceq)$ and $(\mathcal{T}', \preceq')$ complete lattices.
- Supremum $\vee/\vee'$, infimum $\wedge/\wedge'$.
- Dilation: $\delta : \mathcal{T} \rightarrow \mathcal{T}'$ such that

$$\forall (x_i) \in \mathcal{T},\ \delta(\vee_i x_i) = \vee'_i \delta(x_i)$$

- Erosion: $\varepsilon : \mathcal{T}' \rightarrow \mathcal{T}$ such that

$$\forall (x_i) \in \mathcal{T}',\ \varepsilon(\wedge'_i x_i) = \wedge_i \varepsilon(x_i)$$

$\Rightarrow$ All classical properties of mathematical morphology on complete lattices.

Morphological operations using structuring elements

Structuring element: binary relation between two elements.

$$B_x = \delta(\{x\})$$

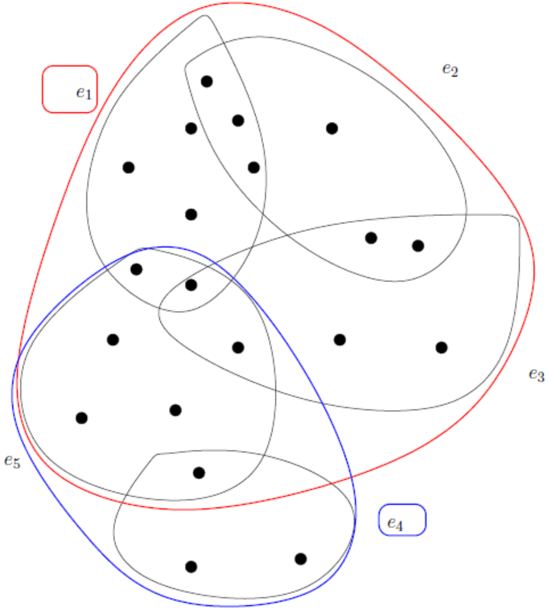
- In $(\mathcal{P}(\mathcal{E}), \subseteq)$: $E = \cup_{e \in E}\{e\}$, and $\delta_B(E) = \cup_{e \in E} B_e = \cup_{e \in E}\delta(\{e\})$.
  Example:

$$\forall e \in E,\ B_e = \delta(\{e\}) = \{e' \in \mathcal{E} \mid V(e) \cap V(e') \neq \emptyset\}$$

- $\delta : \mathcal{T} = (\mathcal{P}(\mathcal{E}), \subseteq) \to \mathcal{T}' = (\mathcal{P}(\mathcal{V}), \subseteq)$
  Example:

$$\forall e \in E, B_e = \delta(\{e\}) = \{x \in \mathcal{V} \mid \exists e' \in \mathcal{E}, x \in e' \text{ and } V(e) \cap V(e') \neq \emptyset\}$$

$$= \cup\{V(e') \mid V(e') \cap V(e) \neq \emptyset\}$$



e_1
e_2
e_3
e_4
e_5

- $\mathcal{T} = (\{H = (V, E)\}, \preceq)$
  - Canonical decomposition (sup-generated lattice):

$$H = (\vee_{e \in E}(V(e), \{e\})) \vee (\vee_{x \in V_{\setminus E}}(\{x\}, \emptyset))$$

  - Example:

$$\forall x \in V_{\setminus E},\ \delta(\{x\}, \emptyset) = (\{x\}, \emptyset) \text{ (for isolated vertices)}$$

$$\forall e \in E,\ \delta(V(e), \{e\}) = (\cup\{V(e') \mid V(e') \cap V(e) \neq \emptyset\}, \{e' \in \mathcal{E} \mid V(e') \cap V(e) \neq \emptyset\})$$

(for elementary hypergraphs associated with hyperdeges)

- Attributed hypergraphs: dilation based on similarity between attribute values.

Dualities

Links between dilations and duality concepts on hypergraphs.

- $H = (V, E)$ hypergraph with $V \neq \emptyset$, $E \neq \emptyset$, $H^* = (V^*, E^*)$ its dual
- Mapping $\delta : V \to \mathcal{P}(V)$
- $\delta^*(x) = \{y \in V; x \in \delta(y)\}$
- $\delta^{**}(x) = \{y \in V; x \in \delta^*(y)\}$
- Theorem:
  a) for all $X \in \mathcal{P}(V)$, $\delta^*(X) = \bigcup_{x \in X} \delta^*(x) = \{y \in V, X \cap \delta(y) \neq \emptyset\}$ (resp. $\delta(X) = \bigcup_{x \in X} \delta(x) = \{y \in V, X \cap \delta^*(y) \neq \emptyset\}$) iff δ^* is a dilation (resp. δ is a dilation);
  b) for all $X \in \mathcal{P}(V)$, if $\bigcup_{x \in X} \delta^*(x) = V$ (resp. $\bigcup_{x \in X} \delta(x) = V$) then $X \subseteq \bigcup_{X \cap \delta^*(y) \neq \emptyset} \delta^*(y)$ (resp. $X \subseteq \bigcup_{X \cap \delta(y) \neq \emptyset} \delta(y)$);
  c) $\delta^{**} = \delta$ on V;
  d) if δ^{**} and δ are dilations then $\delta^{**} = \delta$.

- $H_\delta = (V, (\delta(x))_{x \in V})$
- $\forall x \in V, \delta(x) = \{e \in E ; x \in e\}$
- Theorem: For $H = (V, E)$ without isolated vertex and without repeated hyperedge, $H \simeq H_\delta \Longleftrightarrow H^* \simeq H_{\delta^*}$.

- Theorem:
  - $P = (p_i)_{i \in \{1,2,\dots,t\}}$ discrete probability distribution on V^*, taking rational values $\Rightarrow$ dilation.
  - δ dilation on V^* $\Rightarrow$ discrete probability distribution on V^*.
- Corollary:
  - Discrete probability distribution on V^* $\Rightarrow$ rough space on E (upper approximation = dilation, lower approximation = erosion).
  - Rough space on E $\Rightarrow$ discrete probability distribution on V^*.

$\Rightarrow$ Links between morphological operators, derived rough sets, and probability distributions.

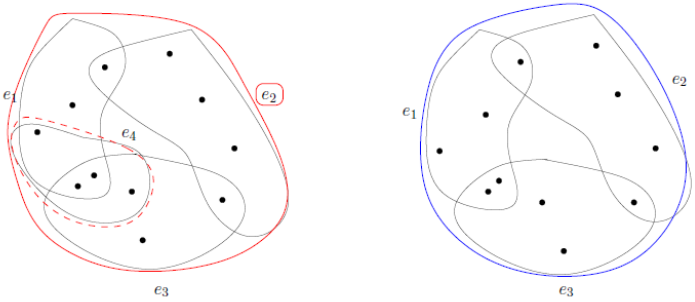
Similarity from dilation

Introducing flexibility in hypergraph comparison, w.r.t isomorphisms.

- $H^1 = (V, E^1)$ and $H^2 = (V, E^2)$ without empty hyperedge.
- δ_{E^1} and δ_{E^2} extensive dilations on E^1 and E^2.
- Example: $\forall A \in \mathcal{P}(E), \delta(A) = \{e \in E; V(A) \cap V(e) \neq \emptyset\}$.
- $\forall (A, B) \in \mathcal{P}(E^1) \times \mathcal{P}(E^2) \setminus (\emptyset, \emptyset)$, similarity:

$$s(A, B) = \frac{|\delta_{E^1}(A) \cap \delta_{E^2}(B)|}{|\delta_{E^1}(A) \cup \delta_{E^2}(B)|}$$

- Proposition:
  a) $\forall (e_i, e_j) \in E^1 \times E^2$, $s((e_i, e_j)) = 0 \Longleftrightarrow E^1 \cap E^2 = \emptyset$;
  b) $\forall (e_i, e_j) \in E^1 \times E^2$, $s((e_i, e_j)) = 1 \Longrightarrow E^1 = E^2$,
  c) s is symmetrical.



- Left (E_1): $\delta(e_1) = \{e_1, e_2, e_3, e_4\}, \delta(e_2) = \{e_1, e_2, e_3\}$...
- Right (E_2): $\delta(e_i) = \{e_1, e_2, e_3\}$, for $i = 1, 2, 3$
- $s(e_1, e_i) = \frac{3}{4}$
- $s(e_2, e_i) = \frac{3}{3}$
- $s(\{e_1, e_2\}; B) = \frac{3}{4}$, for $B \subseteq E_2, B \neq \emptyset$

Conclusion

- New contribution: mathematical morphology on hypergraphs, based on complete lattice structures.
- Links with duality concepts.
- Links with rough sets and probability distributions.

Future work:

- More properties based on morphological operators.
- Exploring similarity concepts.
- Applications in clustering, pattern recognition...