Quasi-linear transformations, numeration systems and fractals

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Introduction

We will define relations between quasi-linear transformations, numeration systems and fractals. A Quasi-Linear Transformation (QLT) is a transformation on $\mathbb{Z}^n$ which corresponds to the composition of a linear transformation with an integer part function. We will first give some theoretical results about QLTs. We will then point out relations between QLTs, numeration systems and fractals. These relations allow us to define new numeration systems, fractals associated with them and n-dimensional fractals. With help of some properties of the QLTs we can give the fractal dimension of these fractals.

Definitions

A = integer matrix, ω = positive integer, $\lfloor\ \rfloor$ = integer part function

Linear transformation : $g : \begin{cases} \mathbb{Q}^n \to \mathbb{Q}^n \\ X \to Y = \frac{1}{\omega}AX \end{cases}$

Definitions

– *Quasi-linear transformation* : $G : \begin{cases} \mathbb{Z}^n \to \mathbb{Z}^n \\ X \to Y = \lfloor\frac{1}{\omega}AX\rfloor \end{cases}$

– *Tile with index* $X \in \mathbb{Z}^n$: $P_X = \{Y \in \mathbb{Z}^n | G(Y) = X\}$

– *Tile of order p with index* $X \in \mathbb{Z}^n$: $P_X^p = \{Y \in \mathbb{Z}^n | G^p(Y) = X\}$

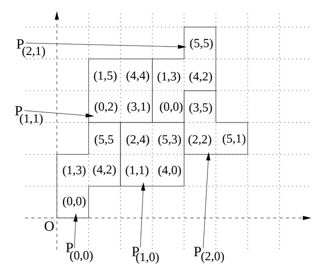

Figure 1: Tiles of a QLT. The index of a tile corresponds to the quotient of an euclidian division, for each point we give the remainder of this division.

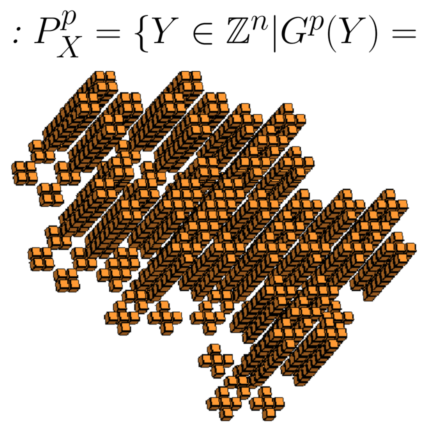

Figure 2: p-tile of the QLT defined by $\frac{1}{3}\begin{pmatrix} -1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & -1 \end{pmatrix}$.

Behaviour under iteration.

Behaviour of the sequence : $X_{n+1} = G(X_n)$

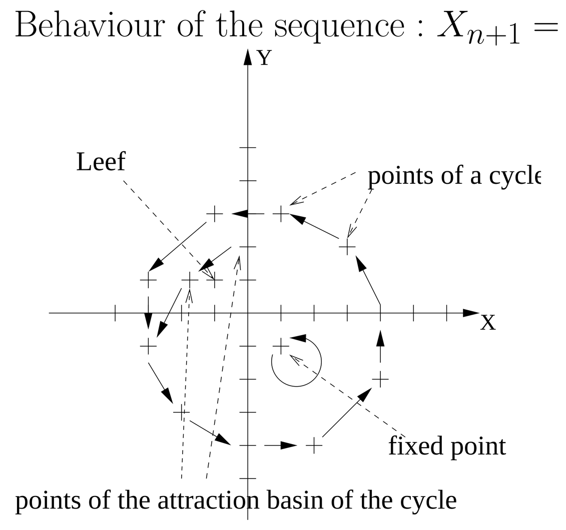

Figure 3: Behaviour under iteration of a QLT

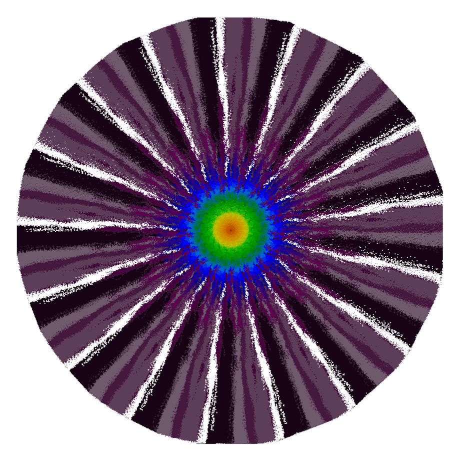

Figure 4: Attraction basins of a QLT with a lot of cycles. The QLT is defined by $\frac{1}{4495}\begin{pmatrix} 4187 & -1622 \\ 1622 & 4187 \end{pmatrix}$.

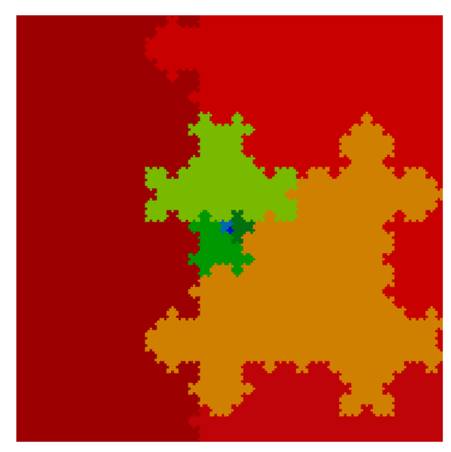

Figure 5: A consistent QLT, the colour of a point depends on the number of iteration necessary to reach O. The QLT is defined by $\frac{1}{3}\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$.

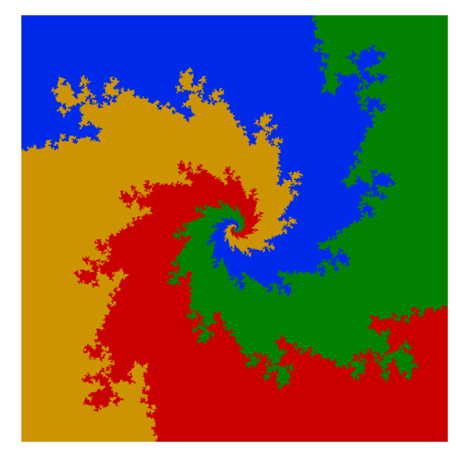

Figure 6: The colour of a point depends on its attraction basins. The QLT is defined by $\frac{1}{5}\begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}$.

Definition *A consistent QLT has the origin O as unique fixed point : For each point Y it exists n such that $(G^n(Y))_{n\geq 0} = O$.*

Theorem *A 2D-QLT G such that $||g||_\infty < 1$ is consistent if and only if one of the three following conditions is verified :*
(1) $b \leq 0,\ a+b \leq 0,\ c > 0$ and $d \leq 0$
(2) $a \leq 0,\ b > 0,\ c \leq 0$ and $c+d \leq 0$
(3) $a \leq 0,\ b \leq 0,\ c \leq 0$ and $d \leq 0$

Tiles associated with particular QLTs

Definition *A QLT defined by $\frac{1}{w}A$ such that $w = m\det(A)$ where m is a positive integer, is called a m-determinantal QLT.*

Proposition *The p-tiles generated by a m-determinantal QLT are all geometrically identical. More precisely, if $\mathcal{T}_v$ refers to the translation of the vector v and if $\widehat{A}^T$ is the transpose of the cofactor matrix of A we have, for all $p \geq 1$:*

$$P_Y^p = \mathcal{T}_{(m\widehat{A}^T)^p Y} P_O^p$$

and $$P_O^{p+1} = \bigcup_{X \in P_O} \mathcal{T}_{(m\widehat{A}^T)^p X} P_O^p = \bigcup_{X \in P_O^p} \mathcal{T}_{(m\widehat{A}^T) X} P_O$$

Proposition *The number of points of a p-tile generated by a m-determinantal QLT in $\mathbb{Z}^n$ is equal to $\delta^{p(n-1)}m^{np}$ where $\delta = \det(A)$.*

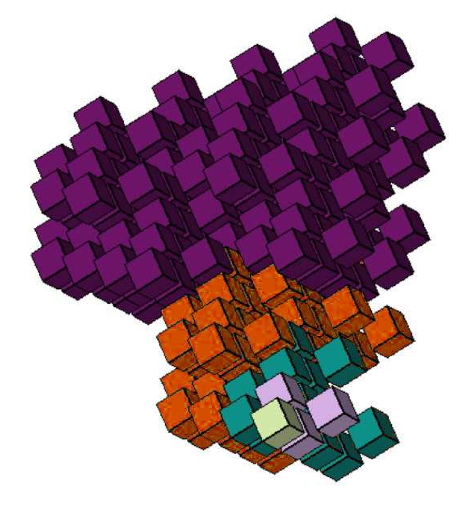

Figure 7: p-tiles of the QLT defined by $\frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$.

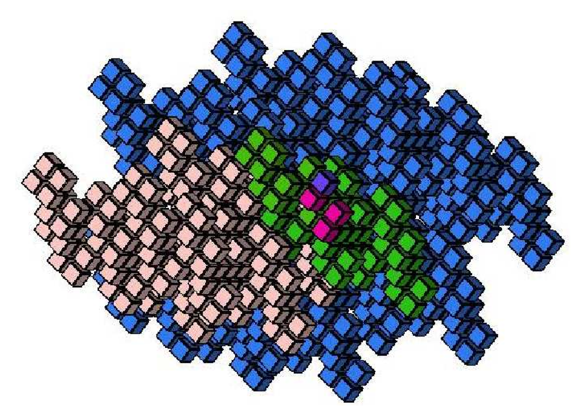

Figure 8: p-tiles of the QLT defined by $\frac{1}{2}\begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & -2 \\ 1 & 0 & 0 \end{pmatrix}$

QLT and numeration systems

Let β denote a complex number and $\mathcal{D}$ a finite set of elements of $\mathbb{Z}[\beta]$. $(\beta, \mathcal{D})$ is a valid base for $\mathbb{Z}[\beta]$ if each element c of $\mathbb{Z}[\beta]$ can be written uniquely in the form $c = c_0 + c_1\beta + c_2\beta^2 \ldots + c_n\beta^n$ with $c_i \in \mathcal{D}$ and $n \in \mathbb{N}$

Gaussian integers

In this section we denote $\beta = a + ib$ with $a, b \in \mathbb{Z}$

Definition *Let $c = x+iy$, the integer division of c by β is defined by :*

$$\left\lfloor\frac{c}{\beta}\right\rfloor = \left\lfloor\frac{ax+by}{a^2+b^2}\right\rfloor + i\left\lfloor\frac{-bx+ay}{a^2+b^2}\right\rfloor.$$

Proposition *Let $c = x+iy$ and $c' = x'+iy' = \left\lfloor\frac{c}{\beta}\right\rfloor$, the point $(x', y') = G_\beta(x, y)$ where G_β is defined by $\frac{1}{a^2+b^2}\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.*

Theorem *Let $\mathcal{D} = \left\{c \middle| \left\lfloor\frac{c}{\beta}\right\rfloor = 0\right\}$, the three following properties are equivalent :*

1. $(\beta, \mathcal{D})$ is a numeration system,

2. The QLT G_β is a consistent Quasi-Linear Transformation,

3. $a \leq 0$ and $|a| + |b| > 1$.

Remark *Note P_O and P_O^p the tiles defined by G_β . P_O^p represents the elements of $\mathbb{Z}[\beta]$ such that the decomposition in the numeration system is of length p.*

Algebraic integers

In this section β denotes an algebraic integer such that $\beta^2 + b\beta + a = 0$ with $a, b \in \mathbb{Z}$. We only consider the case where β is a complex number, that is to say $b^2 - 4a < 0$. Let define $\beta_1 = q + \beta$ with $q \in \mathbb{Z}$.

Definition *Let $c = x' + y'\beta_1$, the quotient of the integer division of c by β is defined by $\lfloor\frac{c}{\beta}\rfloor = \lfloor\frac{x'(q-b)+y'(q^2-qb+a)}{a}\rfloor + \lfloor\frac{(-x'-y'q)}{a}\rfloor\beta_1$.*

Proposition *Let $c = x+iy$ and $c' = x'+iy' = \left\lfloor\frac{c}{\beta}\right\rfloor$, the point $(x', y') = G_{\beta_1}(x, y)$ where G_{β_1} is defined by $\frac{1}{a}\begin{pmatrix} q-b & a-qb+q^2 \\ -1 & -q \end{pmatrix}$.*

Theorem *Let $\mathcal{D} = \left\{c \in \mathbb{Z}[\beta] \middle| \left\lfloor\frac{c}{\beta}\right\rfloor = 0\right\}$, the three following properties are equivalent :*

1. There exists q such that $(\beta, \mathcal{D})$ is a numeration system,

2. There exists q such that the QLT G_{β_1} is a consistent QLT,

3. $(b \geq 2$) or ($b = 1$ and $a \geq 2)$

QLTs and fractals

We consider m-determinantal QLTs and p-tiles associated with them, let define the set

$$K_p = \frac{1}{(m\sqrt{\delta})^p}P_O^p$$

The border of K_p can be generated with a substitution and tends toward a fractal.

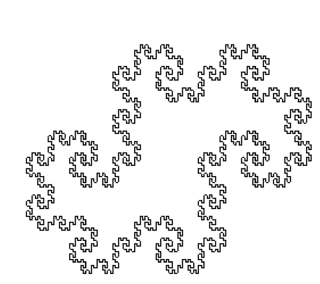

Figure 9: Border of P_O^{12} of QLT defined by $\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$.

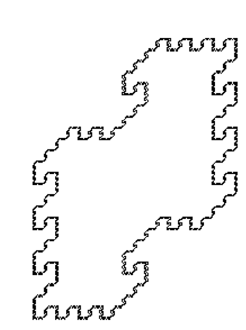

Figure 11: Border of P_O^7 of QLT defined by $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$.

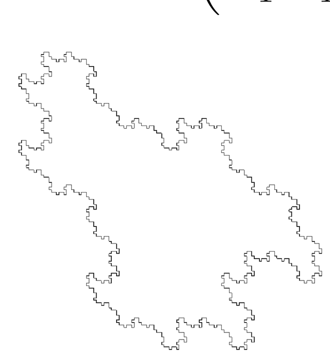

Figure 10: p-tile of the QLT defined by $\frac{1}{3}\begin{pmatrix} -2 & 1 \\ -1 & -1 \end{pmatrix}$.

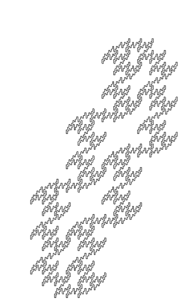

Figure 12: p-tile of the QLT defined by $\frac{1}{3}\begin{pmatrix} -1 & 2 \\ -1 & -2 \end{pmatrix}$.

Denote by P'_O a subset of P_O, and define

$$P'^{p+1}_O = \bigcup_{X \in P'_O} \mathcal{T}_{(m\widehat{A}^T)^p X} P'^p_O = \bigcup_{X \in P'^p_O} \mathcal{T}_{(m\widehat{A}^T) X} P'_O$$

Proposition *Let denote N_p the number of points of P'^p_O and N the number of points removed from P_O to obtain P'_O. We have $N_p = (m^n\delta^{n-1} - N)^p$.*

At each step, we divide the size of the points by $m\delta^{\frac{n-1}{n}}$. If we consider numeration systems, P'_O corresponds to a subset $\mathcal{D}'$ of the set of digits $\mathcal{D}$, so that the fractal obtained corresponds to the set of numbers with zero integer part and whose decomposition uses only the digits of $\mathcal{D}'$.

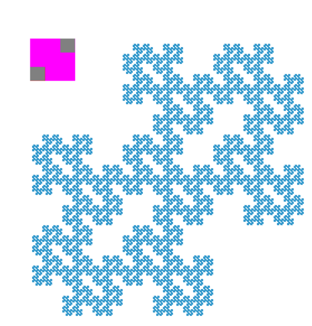

Figure 13: The QLT is defined by $\frac{1}{9}\begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}$, the fractal dimension of the set is $\frac{\log(7)}{\log(3)} = 1,7712$.

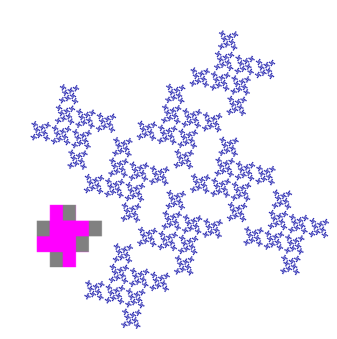

Figure 14: QLT defined by $\frac{1}{13}\begin{pmatrix} -2 & 3 \\ -3 & -2 \end{pmatrix}$, the fractal dimension of the set is $2\frac{\log(8)}{\log(13)} = 1,6214$.

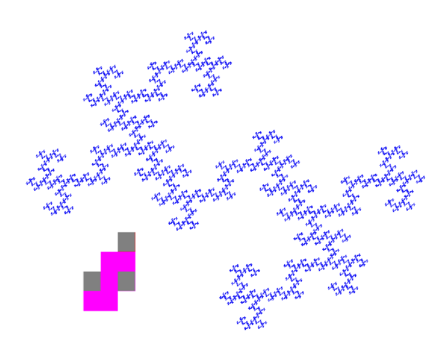

Figure 15: QLT defined by $\frac{1}{9}\begin{pmatrix} 2 & 3 \\ -2 & 1 \end{pmatrix}$ and the fractal dimension of the set is $\frac{2\log(5)}{\log(8)} = 1,5479$.

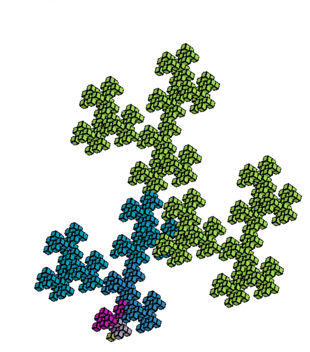

Figure 16: QLT defined by $\frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and the fractal dimension of the set is $\frac{\log(3)}{\log(2)} = 1,585$.